Income Taxation with Elasticity Heterogeneity

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Abstract

How should income taxes account for differences in households’ tax responses? We address this question with a new test that passes if and only if there exists a utilitarian planner for whom the current tax system is locally optimal. Our test takes as inputs standard sufficient statistics, such as the average elasticity of taxable income and the shape of the income distribution, but also incorporates a novel statistic: the variance of elasticities conditional on income. Indeed, the test fails when this variance is sufficiently high. We then proceed to evaluate our test empirically using the NBER panel of tax returns and providing novel estimates of the variance of ETI by income bracket. We find that our optimality test fails, implying there are welfare-improving tax reforms.

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1 Introduction

It is uncontroversial that different people respond to incentives differently. Tax responses are no exception. Indeed, heterogeneity in tax responsiveness is a consistent finding of the empirical literature on the elasticity of taxable income (ETI). In part, this reflects that some households—such as second earners and those near retirement—simply have more elastic labor supply [Blau and Kahn, 2007, Eissa and Liebman, 1996, Vere, 2011]. It also reflects that some households pay more attention to taxes, are more adept at taking advantage of itemizations, or more able to avoid taxes altogether [Taubinsky and Rees-Jones, 2018, Gruber and Saez, 2002, Kopczuk, 2005].

Does heterogeneity in ETIs matter for income taxation? One of the main findings of the literature on taxation with multi-dimensional heterogeneity suggests that it may be unimportant: To first-order, the effects of any tax change on everything a planner values—i.e. tax revenue and the distribution of household welfare—can be computed simply from the mean elasticities at each income level [Werning, 2007, Jacquet and Lehmann, 2015, Scheuer and Werning, 2018]. But beyond the first-order approach, it is less obvious what role heterogeneity may play.

This paper explores the simple but overlooked idea that—unlike the first-order condition explored in the literature—the planner’s second-order condition depends intimately on the structure of household heterogeneity. The second-order condition captures how the effects of an infinitesimal variation in taxes change when one repeats that variation for a second time. Here, the key idea is that each households’ income response to the first tax variation determines at which income level it experiences the second tax variation. For example, if (a) households with the same initial income respond to the first variation differently and (b) the variation differs across their post-first-variation incomes, then repeating the same variation can have different effects on households who faced the same taxes ex-ante.

The advantage of this differential taxation is that—all else equal—the planner prefers to increase taxes on households with lower ETI, whose smaller responses result in smaller tax revenue losses. While the planner cannot explicitly condition taxes on ETI, we show that whenever there is enough heterogeneity in ETI, she can construct a particular tax variation that achieves as if conditioning via a sorting mechanism. Namely, the variation causes low-ETI households to move to incomes where it increases marginal taxes and high-ETI households to move to incomes where it decreases marginal taxes. Repeating the same variation for a second time therefore disproportionately targets marginal taxes on those who respond to them less. In other words, the planner “sorts” households on ETI and then “extorts” them by raising taxes disproportionately more on the unresponsive.

Our main theoretical results encapsulate these ideas in a simple, planner-agnostic “rationalizability test” for the tax-schedule, formulated in terms of locally-observable
sufficient statistics. Our test passes if and only if there exists a social planner within a broad class for whom the tax schedule is optimal among all nearby schedules. The test complements the first-order condition of [Werning 2007] with a novel second-order condition. In contrast to the existing literature, our second-order test depends not only on the mean ETI at each income level, but also on its variance, reflecting the scope for a planner to “sort and extort” households by elasticity.

In light of the well-documented empirical variation in ETIs, this test raises the question: “Is the heterogeneity in ETIs large enough that real world tax schedules can be improved?” The second part of the paper seeks to answer this question by evaluating our rationalizability test in a panel of US tax returns from 1979 to 1990. Doing so requires us to move beyond the existing literature on ETI estimation, which does not study the variance of elasticities conditional on income. We estimate this variance with a number of empirical strategies: First, we provide a lower bound on this variance by comparing mean ETIs of households with different numbers of itemizations. Second, we take a less conservative structural approach that leans on the assumption of linear tax responses. Third, we validate these explicit estimates by providing reduced-form evidence for the sorting mechanism that underlies the failure of the second-order condition. Specifically, we document that income-conditional ETIs change across years as predicted by the differential reshuffling of high- and low-elasticity households in response to observed tax changes.

Our preferred estimates have stark implications for income taxation. We find that—although the first-order test passes—our novel second-order test fails in every year of our sample. This implies that, in our sample, it is impossible to rationalize the income tax schedule within the set of social planners we consider. Said differently, any social planner willing to make at least some minimal welfare trade-offs between households would prefer a different tax schedule. In this sense, a “free lunch” is available through tax reform. A conservative quantification exercise suggests that either raising or lowering top taxes by 20 percentage points results in yearly welfare gains equivalent to approximately $3000 per top earner.

Related Literature Theoretically, we contribute to a long literature on the design of non-linear income taxation schedules when labor is supplied on the intensive margin and production is linear. Recent work by [Werning 2007, Jacquet and Lehmann 2015] and [Scheuer and Werning 2018] has shown that the so-called “ABC formula” of [Diamond 1998]—a necessary first-order condition for optimality—extends to the case of multi-dimensional heterogeneity if one simply uses the average elasticities at each income level.\footnote{Also see a much earlier informal derivation by [Saez 2001].} We complement this result by providing an additional necessary condition—a second-order condition—which, when combined with the first-order condition, is also sufficient.
To our knowledge, we are the first to explicitly study the planner’s second-order condition in a model of non-linear income taxation.\footnote{Werning [2007] shows that when households vary along one dimension and their preferences have a certain functional form, the planner’s problem is concave. This implies that the second-order condition would hold, were one to compute it.}

Rather than fixing a social planner and solving for her optimal tax schedule, we take the tax schedule as given and ask whether there exists some planner (in a broad utilitarian class) whose objective rationalizes it. This approach is most similar to that of Werning [2007], who studies when a given income taxation schedule is Pareto efficient. In both settings, the approach of “testing” rather than “solving” avoids (a) the need to embed strong normative judgments into the positive analysis of tax systems and (b) the need to extrapolate locally-observed sufficient statistics to non-local tax schedules where one does not know their true values. In our setting, the approach has an additional advantage: The version of the second-order condition that guarantees the existence of a planner who rationalizes the tax schedule is dramatically simpler than the second-order condition of a given planner. Despite their similarities, there are two main differences between Werning’s approach and our own. First, whereas he studies a particular one-dimensional environment in which the planner’s problem is globally convex, we work in a general, multi-dimensional environment and study this non-convexity explicitly.\footnote{Werning also discusses how his results apply to settings with multi-dimensional heterogeneity but where the planner may condition taxes on all but one dimension of heterogeneity. This provides an excellent benchmark for our discussion in Section 4.3.1.} Second, our notion of “rationalizability” is very slightly stronger than Pareto efficiency because it requires that (a) the welfare weights a planner assigns to a type are proportional to its measure, and (b) the welfare weights change smoothly enough with each household’s utility.\footnote{We discuss the way this difference affects our results following Theorem 1.}

Also related is the work of Jacquet and Lehmann [2020], who share our interest in multi-dimensional heterogeneity as a reason that ETIs may be endogenous to the tax schedule. However, these authors apply this observation to answer a different and complementary question, namely how heterogeneity shapes optimal non-linear taxes for a Rawlsian social planner. In a similar vein, Lockwood et al. [2020] show how accounting for a planner’s uncertainty over households’ taxable income elasticities affects (a) the ABC formula and (b) optimal taxes. Although different in its motivation, we view Lockwood et al. [2020] as closely related to Jacquet and Lehmann [2020] since—when a planner faces an expected budget constraint—uncertainty over states of the world and heterogeneity across households are interchangeable for the purposes of income taxation.\footnote{This observation, while novel, is not a central focus of our paper. We focus on heterogeneity, but one may interpret our results more generally if the expectations in our rationalizability test taken over households and states of the world simultaneously.} Our more general treatment clarifies the mechanisms introduced by multi-dimensional heterogeneity, especially the interaction between heterogeneity and the shape of the income
distribution. Importantly, neither paper considers the second-order condition of the social planner, so our main focus is quite distinct.

Our results contrast sharply with other recent work providing sufficient conditions for income tax schedules. Namely, the main result of Bierbrauer et al. [2020] implies that, in our setting, the first-order condition of Werning [2007] is not only necessary but also locally sufficient for taxes to be Pareto efficient, even when one allows for multidimensional heterogeneity. By contrast, we show that a distinct second-order condition is necessary in order for taxes to be rationalizable by a planner who values household utility in a smooth way. While our result does not apply to a Pareto planner—since her desire to make transfers to a given household changes discontinuously when that household’s utility crosses a threshold—it does apply to arbitrarily accurate smooth approximations of any Pareto objective. Under any such smooth approximation, the planner becomes willing to tolerate a small welfare loss to any household if it is accompanied by a large enough gain for other households. Our results therefore illustrate a sense in which the Pareto efficiency approach used by Bierbrauer et al. [2020] is knife-edge, falling apart once one allows arbitrarily small trade-offs to be made across households.

On the empirical side, we build on the approach of Gruber and Saez [2002] to estimate the elasticity of taxable income (ETI). We use the same NBER panel of tax returns as these authors as well as a similar identification strategy, identifying ETIs from households’ changes in income following changes in taxes. Our work extends beyond Gruber and Saez [2002] in two ways. First, we estimate elasticities conditional on income level by estimating local polynomials in income space with optimal bandwidths. Second, we place a greater emphasis on the variance of ETI across households. Of course, our interest in ETI heterogeneity has significant precedent in the literature: Gruber and Saez [2002] and Kopczuk [2005] show that itemizers are more elastic, and related work on labor supply elasticities suggests that ETIs may be higher for second-earners, single mothers, and those near retirement. Blau and Kahn [2007], Eissa and Liebman [1996], Vere [2011]. More recent work by Kumar and Liang [2020] emphasizes heterogeneity in average ETIs between income brackets as a key source of variation across ETI estimates in the literature. What distinguishes our approach to the variation in ETIs is an interest in a new, model-implied, sufficient statistic: the variance in ETI conditional on income level. We employ a number

Our work is also related to the multidimensional screening and nonlinear pricing literatures. Whereas the multidimensional screening literature has emphasized direct mechanisms and revelation principle (e.g. Rochet and Stole [2003]), we apply the taxation principle and focus on the indirect mechanism (Hammond [1979], Rochet [1983], Guesnerie [1998]). This approach allows us to sidestep challenging technical issues and derive new conditions for optimality. Our approach can easily be applied to non-linear pricing problems, where our second order condition would translate to a condition for the local convexity of revenues.

The reason is straightforward: if a smooth tax variation that decreases retention at any income level, then it makes earners at that income level worse off to first order, by the envelope theorem.

Although we do not focus on inattention to tax changes, our estimation is related in spirit to Taubinsky and Rees-Jones [2018], who estimate heterogeneity in responses to sales taxes.
of identification strategies tailored to this statistic. Among these strategies is a structural procedure that relies on the linearity of tax responses. We combine this identification strategy with methods from the recent literature on discrete-approximation algorithms in econometrics [Bonhomme and Manresa, 2015, Bonhomme et al., 2017, Lewis et al., 2019].

Outline: The paper is organized as follows: Section 2 presents a simple example that introduces the key ideas of the paper. Section 3 lays out the formal model. Section 4 presents and discusses our theoretical test for rationalizability. Section 5 presents our empirical estimates and evaluates of our test in the data. Section 6 presents a simple quantification of the welfare losses from un-rationalizable tax policy. Section 7 discusses our findings and Section 8 concludes.

2 Motivating example

Before proceeding to the general model, we present a simple example that introduces the key ideas featured in our later formal results.

A unit measure $\mu$ of households $h \in H$ supply labor and consume a good—produced one-for-one with labor—in a static economy. Households face a non-linear tax schedule $T$. Each household $h$’s preferences are additively separable between consumption and labor, feature a constant elasticity of labor dis-utility, and have no income effects:

$$V^h(T) \equiv \max_z z - T(z) - \frac{z^{1 + \frac{\theta^h}{\beta^h}}}{1 + \frac{1}{\beta^h}} / (\theta^h)^{\frac{1}{\beta^h}},$$

(1)

where $\theta^h > 0$ and $\beta^h > 0$ are productivity and elasticity parameters. We let $z^h(T)$ denote the maximizer. Within elasticity groups, productivity is Pareto with a common shape, i.e. $\theta^h | \beta^h \sim \text{Pareto}(\alpha > 1)$.

Now, consider the problem of a planner reforming a constant top tax rate $\bar{\tau}$ that applies to all income earned above income level $\bar{z}$. Let $T_\bar{\tau}$ denote the entire schedule as a function of $\bar{\tau}$. We assume the planner places a constant weight $\tilde{\lambda}$ on transfers to each household in the top bracket, relative to tax revenue, and therefore chooses $\bar{\tau}$ to maximize a weighted sum of welfare and tax revenue in the top bracket, or

$$L(\bar{\tau}) \equiv \tilde{\lambda} \cdot W_{\text{top}}(\bar{\tau}) + \text{Rev}_{\text{top}}(\bar{\tau}) = \mathbb{E}_h \left[ \tilde{\lambda} \cdot V^h(T_\bar{\tau}) + \tilde{\tau} \cdot z^h(T_\bar{\tau}) \mid z^h(T_{\bar{\tau}_0}) \geq \bar{z} \right],$$

(2)

Our assumptions conveniently ensure that $\mathbb{E}_h \left[ z^h(T_\bar{\tau}) \mid z^h(T_{\bar{\tau}_0}) \geq \bar{z}, \beta^h = \bar{\beta} \right] \propto (1 - \bar{\tau})^{\alpha^h}$.

Appendix B.1 walks through the algebra, which is straightforward.

We also assume that (a) the planner is considering a tax increase—to ensure no responses by earners outside of the top bracket—and (b) the tax schedule is convex—which ensures intensive margin responses.
A natural starting place for this analysis is the standard first-order condition for the revenue effects of a small increase in top taxes around its initial level $\bar{\tau}_0$.

$$\left. \frac{d}{d\bar{\tau}} \right|_{\bar{\tau} = \bar{\tau}_0} \mathcal{L}(\bar{\tau}) \propto -\tilde{\lambda} \text{ Welfare effect} + \frac{1}{1 - \bar{\tau}_0} \alpha \mathbb{E}_{\text{top}}[\beta] \text{ Mechanical effect} - \frac{\bar{\tau}_0}{1 - \bar{\tau}_0} \alpha \mathbb{E}_{\text{top}}[\beta^2] \text{ Behavioral effect}$$

where $\mathbb{E}_{\text{top}}[\cdot]$ weights each elasticity proportionally to the total top-bracket earnings of households with that elasticity. The welfare effect captures welfare losses that result—by the envelope theorem—from the increased tax burden at households’ initial incomes. The mechanical effect captures revenue gains as taxes increase at households’ initial incomes, whereas the behavioral effect captures negative fiscal externalities as households reduce their incomes in response to higher taxes.

We later show that in the data, the sum of the two revenue effects is always positive on net: raising taxes increases tax revenues, i.e. we are on the correct side of the “Laffer curve.” Importantly, this implies that there exists a welfare weight $\tilde{\lambda}$ that rationalizes the planner’s first-order condition. To verify that we are not a local minimum, or—as we will say throughout—in a “Laffer valley,” we check the second-order condition:

$$0 \geq \left. \frac{d^2}{d\tau^2} \right|_{\tau = \bar{\tau}_0} \mathcal{L}(\bar{\tau}) \geq \left. \frac{d^2}{d\tau^2} \right|_{\tau = \bar{\tau}_0} \text{ Rev}_{\text{top}}(\bar{\tau}) \propto -(2 - \bar{\tau}_0) \mathbb{E}_{\text{top}}[\beta] + \bar{\tau}_0 \alpha \mathbb{E}_{\text{top}}[\beta^2]$$

Interestingly, this second-order test is sensitive to not only the mean elasticity of top earners, but also its variance $\text{Var}_{\text{top}}[\beta] = \mathbb{E}_{\text{top}}[\beta^2] - \mathbb{E}_{\text{top}}[\beta]^2$. And we are in a “Laffer valley” (i.e. (4) fails) if—fixing the mean elasticity—elasticities vary enough. Moreover, this conclusion is normatively neutral, as the test does not depend on the welfare weight $\tilde{\lambda}$.

What economic mechanism underlies the planner ability to improve taxes when faced with heterogeneous elasticities? This is best illustrated by conceptualizing the second-order effects of a tax increase as the difference between the effects of two successive, infinitesimal tax increases. How does the first tax change shape the effects of the second? Here, the key idea is that the behavioral effects of the first tax change determine at which income level each household experiences the second tax change, and therefore determine the tax change it experiences. For low-elasticity households—who by definition do not respond much to the first tax change—this has little bite. However, many high elasticity households exit the top bracket in response to the first tax change, and so are not affected by the second one. Due to this differential sorting, the first tax change decreases the

\footnote{The second inequality of (4) requires that second-order welfare effects are weakly positive. For the purpose of this example, this relies on our assumption that the planner’s desire to make transfers to each top-earner is constant, not changing as their consumption or income. However, our later results show that this assumption is unnecessary when the planner uses more finely-tuned tax variations.}
effective elasticity of top earners; the planner’s second increase in taxes exploits this.

Figure 1: Income density of high- (blue) and low-elasticity (orange) households before (left panel) and after (right panel) taxes are increased. The compositional shift induced by an increase in taxes reduces the aggregate elasticity of top earners. (The vertical line in the right panel is bunching at a kink.)

Is this theoretical consideration of any practical importance? Consider a back-of-the-envelope calibration: Suppose top marginal tax rates $\bar{\tau}_0$ are 50%, the Pareto shape parameter $\alpha$ is 2.5, and the average elasticity of the top earners is 0.3, consistent with the literature. Then the first-order condition holds if the planner places a weight $\tilde{\lambda} = 0.25$ on transfers to top earners. The second order test (4) holds—so that taxes locally maximize rather than minimize revenue—if and only if $\text{Var}_{\text{top}}[\beta] \leq 0.27$. This is roughly the variance if, for example, three quarters of households have elasticity zero and one quarter have elasticity $\approx 1.2$. This value for the variance is only slightly above a conservative lower bound we estimate in Section 5 based on differences in elasticities between low- and high-itemizers. This suggests that the planner’s second-order condition may well be violated in practice.

The rest of the paper develops the insights illustrated above in order to (a) accommodate general planner and household preferences, (b) provide an efficiency test that applies to the whole tax schedule and is both necessary and sufficient, and (c) formulate this test in terms of sufficient statistics that we then evaluate in the data. The same key ideas—elasticity variance in the planner’s second-order condition and the “sort and extort” motive—continue to play central roles in a general environment with non-linear income taxation.

3 Model

We study a standard, static Mirrlees model of income taxation, but allow for arbitrary preference heterogeneity. After laying out the model, we describe a number of mild regularity conditions used for our main results.
3.1 Environment

Time is static. There is a single consumption good and a single labor factor. Production is linear, and we normalize the price and the wage to one.

A unit measure $\mu$ of households $h \in H$ supply labor $z$ and consume subject to a non-linear income retention schedule to maximize utility $u^h$. For any income retention schedule $\tilde{R} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we define

$$z^h(\tilde{R}) \equiv \arg\max_{z \geq 0} u^h(\tilde{R}(z), z)$$

if the arg max exists, and let $V^h(\tilde{R})$ denote the associated max.

We take as a primitive a particular income retention schedule $R : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that maps each household’s pre-tax income to its post-tax income, i.e. consumption. We denote labor supply, consumption, and utility at $R$ by $z^h_0$, $c^h_0$, and $V^h_0$, respectively. Throughout, we will use “income retention schedule” and “tax schedule” interchangeably.

Throughout the paper, we study tax changes $\Delta : \mathbb{R}_{\geq 0} \to \mathbb{R}$ relative to $R$. Our main results focus in particular on very “small”—i.e. local—tax changes. Put simply, we call a tax change small if its level and derivatives are uniformly bounded by those of $R$, with a small bound. More formally, we define a norm on the space of tax changes\(^\text{12}\) by

$$||\Delta|| = \sup \left\{ B \in \mathbb{R}_{>0} \mid \forall z \in \mathbb{R}_{>0}, |\Delta(z)| \leq B|R(z)| \text{ and } |\Delta'(z)| \leq B|R'(z)| \right\}. \quad (7)$$

We denote by $0$ the zero tax change and by $B_\delta(0)$ the open ball of radius $\delta$ around $0$.

3.2 Class of social planners

Throughout the paper, we study the problem of hypothetical social planners who assess welfare according to “generalized utilitarian” criteria and face tax revenue constraints.

**Definition 1.** We call a level of government expenditure $G$ and a collection of welfare-weighting functions $(w^h)_{h \in H}$ with $w^h : \text{Im}(u^h) \to \mathbb{R} \cup \{-\infty\}$ a **standard social objective** if there exists $\delta > 0$ such that, for all $h \in H$, $w^h$ satisfies the following on $V^h(R + B_\delta(0))$:

- $w^h$ is finite.

\(^{12}\)Our regularity assumptions ensure the derivatives used in this definition exist. In Appendix A.1 we define a Banach space of tax deviations on which $||\cdot||$ is defined and is a norm in the formal sense.
• $w^h$ is twice-continuously differentiable,
• $w^h$ is weakly increasing, and strictly increasing for a positive measure of $h \in H$;
and $w^h \circ V^h(\tilde{R})$ and its first two Frechet derivatives in $\tilde{R}$ are $h$-measurable within $R + B_δ(0)$.

Our main results rely on the notion that a social objective may "rationalize" the tax schedule $R$ by providing a metric according to which it is optimal among all other (nearby) tax schedules.

**Definition 2.** A standard social objective $((w^h)_{h \in H}, G)$ **locally rationalizes** $R$ if there exists $δ > 0$ such that

$$R \in \arg \max_{\tilde{R} \in R + B_δ(0)} \int w^h \circ V^h(\tilde{R}) \quad \text{s.t.} \quad \int [z^h(\tilde{R}) - \tilde{R}(z^h(\tilde{R}))] \, d\mu \geq G.$$  

The regularity conditions we impose for our formal results ensure that aggregate welfare and tax revenue always exist locally to $R$, so (8) is well-posed.

By working within the class of standard social objectives, we restrict our focus in a few important ways. First, the importance of any set of households to welfare is proportional to their measure. Second, welfare is additively separable across households; this rules out certain explicit forms of fairness concerns. Third, welfare is increasing in each household’s utility. Finally, welfare is sufficiently smooth in each household’s utility.

The class of standard social objectives nests most, but not all planners considered in the income taxation literature. One notable exception is a “Pareto planner” who prioritizes getting each household’s utility above a (household-specific) “target utility.” A Pareto planner’s marginal desire to allocate additional utility to any particular household is discontinuous at its target utility for that household, whereas a planner with a standard social objective values this additional utility in a continuous way. Another exception is a “Rawlsian planner” who maximizes the utility of the worst-off household. Although such a planner lacks a standard social objective due to non-separability, one may think of the Rawlsian motive as a limit case of standard social objectives with increasingly concave weighting functions $w^h$.

### 3.3 Regularity conditions

Whereas many results in the optimal income taxation literature rely on specific function form assumptions, our theoretical results instead hold under much weaker regularity conditions. For readability, we summarize the main context of these assumptions below and state them formally in Appendix A.

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13These derivatives exist by $w^h$'s differentiability, Assumption 2 and (for small enough $δ$) Lemma 2; see Appendix A.
Our regularity conditions serve four main purposes. First they ensure that welfare and tax revenue are well-defined and well-behaved (i.e. twice-continuously differentiable) in a neighborhood around the initial tax schedule $R$. The main conditions we impose to this end are that the initial tax schedule $R$ is sufficiently smooth and has bounded elasticities, that—in a similar spirit—household utility is sufficiently smooth and generates income and compensated elasticities of labor supply that are bounded across households.

Second, our assumptions ensure that welfare and tax revenue can be expressed as integrals over household income $z$, facilitating integration by parts. To this end we assume the existence of sufficiently smooth functions for the income density $g(z)$ and income-conditional elasticity moments (e.g. the mean compensated elasticity of labor supply conditional on income $z$).[14]

Third—and of somewhat more qualitative interest—we assume that, local to the initial tax schedule $R$, all labor is supplied on the intensive margin. While this assumption is in line with many core models of income taxation, it abstracts away from interesting extensive margin decisions, such as labor force participation and migration. We view the extensive margin of labor supply as an interesting area for future work.

Fourth, we work with a slightly stronger notion of a standard social objective, which we call a **standard, regular social objective**. This ensures the existence of income-conditional moments for several welfare statistics—such as the planner’s desire to transfer income to households at income $z$—which are sufficiently smooth in income. Our results use the assumption that average income-condition welfare weights are continuous in income in order to argue that a small decrease in the welfare of households earning $z$ can be fully compensated (from the perspective of aggregate welfare) by a proportional increase in the welfare of $z + dz$ earners.

## 4 Rationalizability test for tax schedules

In this section, we present our main theoretical results. We provide a set of simple conditions on locally-observable sufficient statistics—i.e. a “test”—that holds if and only if the tax schedule is locally rationalizable. Our test augments the first-order test of [Werning 2007] with a novel, second-order condition.

After showing these conditions are necessary for rationalizability, we offer two interpretations: First, we illustrate variational arguments about the first- and second-order changes in welfare and tax revenue that tax change. Second, we explain the economic mechanisms at work, drawing particular attention to a novel “sort and extort” motive for taxation with multi-dimensional heterogeneity. Finally, we show that the conditions

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14While the necessity of our rationalizability test only relies on weak assumptions of this form, the sufficiency of our test additionally requires some mild conditions on the relationship between taxes, incomes, and conditional elasticity moments in the limits of zero and infinite incomes.
are also locally sufficient for rationalizability.

4.1 Necessary conditions

In order to state our first result, we will now introduce notation used throughout the paper. For all \( z \in \text{supp} \ g \), we denote by \( \eta(z) \), \( \varepsilon(z) \), and \( \alpha(z) \) the average income and compensated elasticities\(^{15}\) and the local shape parameter of the income distribution at income \( z \)\(^{16}\)

\[
\eta(z) \equiv \mathbb{E}[\eta^h(R)|z_0^h = z], \quad \varepsilon(z) \equiv \mathbb{E}[\varepsilon^h(R)|z_0^h = z], \quad \alpha(z) \equiv -\frac{d \log (zg(z))}{d \log z} \tag{9}
\]

Income elasticities capture behavioral responses to changes in the level of income retention, whereas compensated elasticities capture behavioral responses to changes in the slope of income retention. In addition to these familiar statistics, our results also depend on two less well-studied statistics. These are the income-conditional average compensated elasticity squared and an income-conditional average compensated super-elasticity \( \varepsilon^+(R) \) that captures how households’ compensated elasticity change through to their labor supply responses to marginal tax changes\(^{17}\) We use the notation

\[
\varepsilon^2(z) \equiv \mathbb{E}[\varepsilon^h(R)^2|z_0^h = z], \quad \varepsilon^+(z) \equiv \mathbb{E}[\varepsilon^{+h}(R)|z_0^h = z]. \tag{10}
\]

Our first main result provides necessary conditions that constrain the relationships between these parameters and taxes at any locally rationalizable tax schedule. This test consists of a first order condition, which is standard in the literature, and a complementary second order condition, which is novel.

**Theorem 1.** Suppose \( R \) is locally rationalized by some standard, regular social objective. Then for all \( z \in \text{supp} \ g \),

\[
0 \geq -1 + (1-R'(z))z \left[ \frac{\eta(z)}{R(z)} + \left( \frac{\alpha(z) - \frac{d \log}{d \log z} \left( \frac{1-R'(z)}{R'(z)} \right) - \frac{d \varepsilon(z)}{d \log z} \right) \frac{\varepsilon(z)}{R'(z)z} \right]. \tag{ABC}
\]

Moreover if \( \text{(ABC)} \) does not everywhere hold with equality\(^{18}\) then for all \( z \in \text{supp} \ g \),

\[
0 \geq - (1+R'(z))z \varepsilon(z) + (1-R'(z))z \left[ \alpha(z)\varepsilon^2(z) - \frac{d \varepsilon^2(z)}{d \log z} + \varepsilon^+(z) \right]. \tag{DEFG}
\]

**Proof.** See Appendix \( \text{C.1} \). \( \square \)

\(^{15}\)We define these elasticities more formally in Appendix \( \text{A.2} \).

\(^{16}\)This is related to the well-known concept of the income distribution’s “Pareto tail.” The local shape parameter is a local version of this concept that applies to all functions and coincides with the Pareto shape parameter if \( g \) is the density of a Pareto distribution.

\(^{17}\)See Appendix \( \text{A.4.1} \) for a more formal definition.

\(^{18}\)(ABC) does holds everywhere with equality if and only if \( R \) is a stationary point for tax revenue; this is generically false.
We have labelled the first-order condition “ABC” in analogy to Diamond [1998]’s well-known “ABC formula”, because they both derive from the planner’s first-order condition. More precisely, \( (ABC) \) can be understood as a differentiated version of Diamond’s formula. Analogously, we dub our second-order test “DEFG.”

The first-order condition \( (ABC) \) and the second-order condition \( (DEFG) \) reflect two very different senses in which \( R \) is robust to changes in taxes. First, \( (ABC) \) says that the planner cannot lower taxes locally to \( z \) without reducing tax revenue, i.e. we are on the correct side of the Laffer curve. If this condition fails then—since it is possible to lower taxes at the same time as raising tax revenue—taxes are Pareto inefficient [Werning, 2007]. By contrast, a failure of the second-order test does not imply Pareto inefficiency but rather a slightly weaker condition: non-rationalizability within the class of planners we consider. The distinction is that, whereas the planner responds to a failure of \( (ABC) \) by only lowering taxes, the planner responds to a failure of \( (DEFG) \) at \( z \) with a small decrease in taxes just above \( z \) and an equal and opposite increase in taxes just below \( z \).[19]

Although this makes some households just below \( z \) worse off, we show that their decrease in welfare can be made arbitrarily small relative to the planner’s revenue gains. This willingness to make welfare trade-offs—so long as the downside is small enough—is the main feature that distinguishes our notion rationalizability from Pareto efficiency.

Another important contrast between the standard first-order condition and our new second-order condition is that the former only depends on average elasticities within each income level, whereas the latter depends on a higher moment of the distribution of elasticities. Critically, in order to evaluate \( (DEFG) \), one must—since \( \varepsilon^2(z) = \varepsilon(z)^2 + \text{Var}[\varepsilon_h(R)|z_h = z] \)—consider the variance of compensated elasticities within each tax bracket. For this reason, the inclusion of multi-dimensional heterogeneity can have profound effects on the planner’s problem. Namely, when the income density is falling quickly enough \( (\alpha(z) > 0) \)—as at high income levels—the planner’s second-order condition fails whenever the variance in elasticities is sufficiently large. In this case, it is not possible to rationalize the tax schedule \( R \) within the broad class of social objectives we consider, even though the first-order \( (ABC) \) test known to the existing literature may pass.

4.2 Variational interpretation

In this section, we explain the first- and second-order conditions of Theorem 1 by illustrating the variations in taxes that they reflect. Intuitively, if \( R \) is rationalizable—i.e.

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[19] The reason we use this particular deviation is that a planner’s second-order condition is only informative in directions in which their Lagrangian is constant to first order (other directions are either infeasible or welfare-reducing). Our assumptions guarantee that the Lagrangian is constant to first order under this deviation because they guarantee that the marginal effects of tax changes around some income level—both on household behavior and on welfare—are continuous both in (a) what that income level is and (b) the sign of the tax changes.
optimal for some social planner—then any variation in taxes must have a zero first-order effect and a negative second-order effect on that planner’s Lagrangian.

To begin, consider the first-order effects of a change in taxes that smoothly increases retention between \( z - dz \) and \( z + dz \), as in Figure 2. These effects can be decomposed, à la Saez [2001], into welfare effects, mechanical revenue effects, and behavioral revenue effects. Welfare effects simply reflect that—since we have weakly raised retention everywhere—households are better off as a result of the tax change. Mechanical revenue effects capture the losses in tax revenue that occur because, ignoring any change in household labor supply behavior, taxes are lower at each income level. Finally, behavioral revenue effects reflect that tax-change-induced changes in household behavior—that through both income and compensate effects—have fiscal externalities on the planner whenever marginal taxes are non-zero.

Since the welfare effects are positive, revenue effects must be negative at any rationalizable schedule (otherwise the tax schedule can be improved). The RHS of (ABC) captures these revenue effects: The \(-1\) corresponds to mechanical revenue losses and the remaining terms correspond to behavioral effects, which can increase or decrease revenue.

We now consider the second-order effects of the same variation in taxes we have already studied to first order. The second derivative of the planner’s Lagrangian considers how its first derivative changes as the same variation is done more or less. We therefore consider, “how do the effects of an infinitesimal variation in taxes differ when we do it for a second time?” If \( R \) is optimal for some planner, then these second-order effects must be negative on net—i.e. the Lagrangian must be locally concave—in order to guarantee that there is not an improvement in the tax schedule. Our second-order test (DEFG) computes these effects, and, in analogy to (ABC), organizes them represents changes in welfare effects.

Figure 2: Left panel: An initial income retention schedule, before (solid) and after (dashed) a small variation in taxes. The shaded blue area represents the size of the mechanical effect and (if appropriately weighted) the welfare effect. Right panel: Behavioral effects due to a the same variation in taxes. Behavioral income effects (black arrow) act on households whose level of income is affected (shaded orange area), and can have positive or negative sign. Behavioral compensated effects increase the incomes of households whose marginal taxes decrease (green), and vice-versa (red).
with an inequality and expresses changes in revenue effects explicitly on the RHS. Since (as we argue below) the change in welfare effects is—for any planner—positive, so the change in revenue effects must be negative. This explains the basic structure of \( \text{DEFG} \) and why it is possible to express without taking a stance on the planner’s welfare weights. Below, we explain each of its terms in more detail.

Before doing breaking down the terms of our second-order condition, let us make one clarification: As discussed below Theorem 1, our second-order condition is derived by considering a small increase in retention and a small, nearby decrease in retention.\(^{20}\) Below, we keep things simple by discussing the second-order effects of only the increase in retention.\(^{21}\)

We now turn to the tax variation and its second order effects. We begin with a restricting our attention to a particular class of tax variations: those which are “narrow” in the sense that the amount they change retention at their peak is much greater than the range of incomes over which they change retention. Crucially, a typical household affected by a narrow tax change experiences a much greater change in marginal taxes than in the level of taxes. This simplifies our analysis in two main ways. On the welfare side of the planner’s problem, it allows us to ignore second-order welfare effects related to changes in the planner’s desire to transfer more income to a household to whom they have already made some transfers. On the revenue side of the planner’s problem, it allows to ignore income effects and focus solely on behavioral effects.\(^{22}\)

We now walk through the second-order effects of a narrow tax variation concentrated between \( z - dz \) and \( z + dz \), asking “how do its effects differ when we do it for a second time?” We begin with the welfare and mechanical effects, which go hand in hand. For a narrow tax variation, the only second-order effect on welfare is as follows: When taxes change for a second time, the amount of post-tax income transferred to each household may differ the amount transferred during the first tax change, since—due to its behavioral response to the first tax change—that household’s income is different than it was during the first tax change. Since these first-order behavioral effects shift all households with incomes in \( [z - dz, z + dz] \) toward \( z \), where the change in retention is higher, the second tax variation transfers more post-tax income to each household than did the first; see Figure 3. For this reason, the second-order effect on welfare is positive. The second-order mechanical revenue effect is just the flip side of the same coin: Because the second tax variation transfers more post-tax income to each household, it costs more for the planner. So the second-order welfare effect is positive, while the second-order mechanical revenue effect is negative; this accounts for the \( \geq \) sign and a factor of \(-R'(z)\varepsilon(z)\) term.

\(^{20}\)Performing both simultaneously ensures that the planner’s Lagrangian is constant to first order.

\(^{21}\)We show in the proof of Theorem 1 that this is without loss because although the decrease in retention has opposite first-order effects, it has identical second-order effects.

\(^{22}\)Formally, we do not “ignore” these effects. Rather, our proofs explicitly model them but show that, for sufficiently “narrow” tax variations, they are dominated by other considerations.
We now shift our focus to the second-order behavioral revenue effects of a narrow tax variation. These can be divided into three groups: (a) two effects proportional to elasticities, (b) one key effect proportional to elasticity-squared, and (c) two small “correction” terms. We begin with the two elasticity-proportional effects: First, the second tax variation generates smaller fiscal externalities for given changes in behavior, since the first tax variation has lowered marginal taxes where households increase labor supply and raised marginal taxes where households decrease labor supply. This contributes a factor of \(-R'(z)z\varepsilon(z)\) to (DEFG). Second, the following mechanism leads to less revenue-advantageous changes in behavior: Where the first variation has increased marginal retention, households respond less to the second tax change since the change in log marginal retention is smaller\(^{23}\) and vice-versa. This dampens positive labor supply responses and amplifies negative labor supply responses, contributing a factor of \(-(1 - R'(z))z\varepsilon(z)\).

We now turn our attention to a key effect of interest, one proportional to the average square of elasticities at \(z\). The dependence of this effect on a higher moment of the elasticity distribution means that it—unlike the previous effects discussed—can have a qualitatively different effect when elasticities vary conditional on income. The effect is as follows: Because households adjust their incomes in response to the first tax change, they experience the second tax change at a different income level, where it may be larger or smaller. Moreover, how many households relocate to each different income level—and so how many households experience greater or smaller increases in taxes during the second tax change—depends on how many households start at each income level, i.e. on the local shape of the income distribution. For example, suppose the income density is sharply decreasing locally to \(z\). In this case, the amount of income earned by households who—due to their behavioral responses to the first tax change—face larger marginal tax decreases or smaller marginal tax increases during the second tax change than the first is greater than the income earned by households who face smaller tax decreases or larger tax increases during the second tax change. This is because the former set of households are those inhabiting \([z - dz, z - \frac{dz}{2}]\) and \([z, z + \frac{dz}{2}]\), which contains more mass than the complementary range of incomes when the income density is sharply decreasing; see, e.g. Figure 3. Since the larger first group responds with more advantageous behavioral effects to the second tax change than the first—whereas the second group does the opposite but is smaller—the second tax change has a more positive behavioral effect on revenue than the first. If, on the other hand, the income density is decreasing around \(z\), the effect flips. In total, this effect accounts for the term \((1 - R'(z))z\alpha(z)\varepsilon^2(z)\) in (DEFG). \(\alpha(z)\) reflects the shape of the income density locally; \(\varepsilon^2(z)\) reflects that households’ elasticities determine not only how much they relocate different income levels (and so different tax changes) but also how strongly they respond to those changes.

\(^{23}\)This is true to the extent that household elasticities remain constant; see later discussion.
Finally, two additional “corrections” to these second-order behavioral effects account for the final two terms of (DEFG). We emphasize these terms less because—to the extent we can estimate them—they are quantitatively unimportant, and because they are zero in many parameterizations used in the literature. The first adjusts the effect described in the previous paragraph to account for the possibility that the distribution of elasticities conditional on income may vary with income locally to $z$. The second captures the fact that, due to changes in the curvature of household preferences and/or the tax schedule, households may have different elasticities during the second tax change than during the first. To the extent we can estimate these effects, they appear to be small (see Section 5.3.2). However, the estimation of so-called “super-elasticities”—which represent changes in elasticities due to changes in the curvature of household preferences—is beyond our scope, and we believe it is an interesting area for future work.

4.3 Exploring the economic mechanism

While the analysis of Section (4.2) explains the various terms of our second-order test (DEFG), we would also like to understand the economic forces that underlie it. Indeed, whether test can fail in any reasonable description of the economy is an important question, considering that Werning [2007] shows the first-order condition (ABC) is both necessary and sufficient for global Pareto efficiency in a certain one-dimensional setting.

In this section, we show under what conditions (DEFG) fails and, when it does, provide an intuitive explanation of how the planner can improve the tax schedule. Our main focus is on the way that the test fails when—due to multi-dimensional household heterogeneity—elasticities vary widely within income levels. Here, we emphasize a novel “sort and extort” procedure through which the planner can increase tax revenue by using
a first tax change to shift the elasticity-composition of the income distribution and a second tax change to exploit the separation of high- and low-elasticity households with differential taxes. Finally, we also explain how the test can fail even when households differ along single dimension of heterogeneity (i.e. productivity).

### 4.3.1 Multi-dimensional heterogeneity: “sort and extort”

The most natural and empirically relevant case in which (DEFG) fails is when there is significant variation in ETIs within income levels. Since one dimension of heterogeneity in household preferences is required to rationalize income differences, within-income differences are only possible with multi-dimensional heterogeneity. These within-income differences in household ETI point to an important limitation faced by the social planner. In general, a social planner might like to tax different households with the same income differently, either because she prefers some households to others or because she anticipates that they may respond to taxes differently. For example, a revenue-maximizing planner levies high taxes on low-ETI households and low taxes on high-ETI households, all else equal. However, when she cannot condition taxes explicitly on these elasticities, the planner is forced set a single tax schedule that—at each income level $z$—must balance her desire to tax different $z$-earners differently. In this sense, the planner’s ability to target taxes is limited by the fact that her tax instrument is lower-dimensional than the space of households she taxes with it.

This constraint—that the planner must set a single tax schedule for households who vary within income levels—is at the core of failures of the second-order condition. To see why, consider the simple case where households can be partitioned into a finite set of groups $i \in I$, each satisfying our regularity conditions in isolation. We show in Appendix C.3 that in this case the aggregate first- and second-order tests are simply the averages of the within-group first- and second-order tests. That is, if we let $\Pi_i^{ABC}(z)$ and $\Pi_i^{DEFG}(z)$ represent the right-hand sides of (ABC) and (DEFG) but when expectations are taken only over households within each group $i$, then the our test can be expressed, for each $z \in \text{supp } g$ as:

$$\mathbb{E} \left[ \Pi_i^{h(z)}_{ABC}(z) \mid z_0^h = z \right] \leq 0 \quad \text{ and } \quad \mathbb{E} \left[ \Pi_i^{h(z)}_{DEFG}(z) \mid z_0^h = z \right] \leq 0.$$  \hfill (11)

Moreover, if preferences within each group $i$ are concave and additively separable over consumption and labor and vary only in a labor dis-utility shifter, then—consistent with Werning [2007]’s observation that the Pareto problem is convex with one such group—one may show that:

$$\Pi_i^{ABC}(z) \leq 0 \quad \Rightarrow \quad \Pi_i^{DEFG}(z) \leq 0.$$  \hfill (12)

\footnote{See Appendix C.3}
Putting together these two observations, we note that the second-order test can only fail if the first-order test fails for at least one group. In other words, all failures of \((\text{DEFG})\) can be attributed to the fact that for at least one group, \(R\) is on the wrong side of the Laffer curve.

In a world with group-specific taxes, the planner could address the fact that one group is on the wrong side of the Laffer curve by simply lowering its taxes. However, this option is not available to the planner we consider: Any desirable decrease in taxes for one group requires a (potentially) undesirable decrease in taxes for another group. Indeed, if group-specific failures of the first-order condition are weak enough, it is possible for the second-order condition to hold still and for taxes to be rationalizable. In this context, the key insight of Theorem 1 is as follows: If group-specific failures of first-order conditions are so severe that the aggregate second-order condition fails, then the planner can improve taxes with a special deviation that accomplishes “as-if” group-specific taxation.

The rest of this section explains how the tax variation underlying \((\text{DEFG})\)—the one explored more mechanically in Section 4.2—approximates group-specific taxation. A first step is to recall that since this tax variation is arbitrarily “narrow”, it has arbitrarily small welfare effects and arbitrarily small revenue effects through income elasticities. This implies that its impacts all operate through compensated elasticity effects. In particular, the sense in which our tax variation approximates group-specific taxation is by effectively lowering taxes more for high-elasticity households and raising taxes more for low-elasticity households.

Our tax variation accomplishes this “as-if” elasticity-dependent taxation with a two-step change in taxes, captured by the first and second derivatives of the planner’s Lagrangian. To first-order, changing taxes causes households to change their labor supply behavior as depicted in Figure 3. Crucially, low-elasticity households end up with roughly the same income density after an infinitesimal change in taxes, whereas high-elasticity households end up with a very different density, one which depends both on their initial density and the shape of the tax change itself. Most importantly, when the initial density is declining steeply, high-elasticity households are disproportionately drawn toward incomes where the variation decreases taxes by more or increases taxes by less, as explained in Section 4.2. That is, the variation’s first-order effect is to “sort” households differentially, by elasticity. The second derivative of the planner’s Lagrangian asks how the first-order effects change when the variation is performed a second time. Critically, these effects improve as the variation is performed a second time, because they now operate on households who have been sorted in just the right way to generate

\[\text{These compensated elasticity effects are—as illustrated for the first order variation at the beginning of Section 4.2—moderated by the local shape parameter } \alpha(z). \text{ Although our test accounts for cross-group differences in this shape, we keep this discussion simple by assuming that the shape parameter is constant across groups } i.\]

\[\text{Because the variation is narrow, these effects can be simplified to compensated effects on revenue.}\]
positive behavioral effects on revenue: High elasticity-households are disproportionately represented where taxes decrease and low-elasticity households are disproportionately represented where taxes increase. That is, the planner exploits the sorting of households by “extorting” only the least-responsive households with high taxes, as if she could tax them differentially.

4.3.2 One-dimensional heterogeneity: “shift and exploit”

Although the most economically interesting and empirically plausible failures of (DEFG) derive from within-income household heterogeneity, such failures are also possible in models where households are homogeneous conditional on income. The main idea behind this possibility is similar to the “sort and extort” mechanism explored in the previous section: The planner can change elasticities at each income level with a first tax variation and then exploit this change by repeating the variation. The difference relative to the heterogeneity case is that elasticity changes come not through sorting but rather through “shifts” – either in the elasticity of individual households or in the identity of which households (each having fixed elasticities) are represented at each income level.

Notably, it is impossible for a planner to “shift and exploit” households when utility satisfies the functional form studied by Werning [2007]. This functional form implies that all variation in compensated elasticities across income levels is due to differences in income and consumption levels at which households’ preferences are evaluated, rather than differences in the curvature of their preferences at a given level of income and consumption. It therefore prevents the planner from using a small tax variation to shift local elasticities, since—for any income level \( z \)—the households who each \( z \) after a tax change have the same post-tax change elasticity as the household who previous earned \( z \) had before the tax change. We provide explicit examples and more detailed discussion in Appendix B.2.

4.4 Sufficient conditions

So far, we have provided a novel set of necessary conditions that must be satisfied by any locally rationalizable tax schedule. However, we derived the second-order condition of our test using only very “narrow” variations in taxes. So tax schedules which pass the (DEFG) test might still allow for other sorts of improvements. Our second main result says that this is not the case: The necessary conditions of Theorem 1 are also sufficient for local rationalizability.

This sufficiency result relies on a few mild assumptions beyond those required for Theorem 1. Aside from the additional regularity conditions in Assumption 6 and the

\[27\] This functional form includes additive CES preferences, which is perhaps the most commonly analyzed example in the literature.
condition—stated in the theorem itself—that some households at each income level have small enough elasticities, we also require that (ABC) and (DEFG) hold in a slightly stronger sense. Finally, our sufficiency result uses a slightly weaker notion of local optimality than the necessity result.\footnote{Even without this weaker notion, there exists a welfare function that, for any $\Delta \in \Delta$, prefers $R$ to $R + \epsilon\Delta$ for all $\epsilon > 0$ below some $\epsilon_\Delta > 0$. The role of our weaker notion of local optimality is to slightly restrict $\Delta$ in a way that ensures $\epsilon_\Delta$ is uniform across all remaining tax changes $\Delta$.}

We can now present Theorem 2, which shows that our characterization of locally rationalizable tax schedules in Theorem 1 is tight: If (ABC) or (DEFG) fail, then the schedule sub-optimal for all planners; if they pass then the schedule is optimal for some planner.

**Theorem 2.** Suppose that

- (ABC) and (DEFG) hold by amounts $\Pi_{ABC}(z)$ and $\Pi_{DEFG}(z)$ that satisfy:
  - For all $z \in \text{supp } g$, $\Pi_{ABC}(z) > 0$.
  - There exist $\bar{b}_c, \bar{b}_z > 0$ such that $\Pi_{DEFG}(z) \geq \bar{b}_c R(z) + \bar{b}_z z$ for all $z \in \text{supp } g$.
  - There exists $M$ such that $z|\Pi'_{ABC}(z)| \leq M|\Pi_{ABC}(z)|$ for all $z \in \text{supp } g$.

- For some sufficiently small $\epsilon > 0$, $\mathbb{P}[\epsilon^h(R) \leq \epsilon | z_0^h = z] > 0$ for all $z \in \text{supp } g$.\footnote{More formally, we assume there exists a conditional expectation function $p_\leq(z; \epsilon) \equiv \mathbb{E}[1_{\epsilon^h(R) \leq \epsilon | z_0^h = z}]$ such that for all $z \in \text{supp } g$, $p_\leq(z; \epsilon) > 0$. A conditional probability function exists because $z_0^h$ is measurable by Assumption 3, because—since $\epsilon^h(R)$ is measurable (see the second-to-last step in the proof of Lemma 2)—the indicator $1_{\epsilon^h(R) \leq \epsilon}$ is measurable, and because this indicator is also therefore integrable by dominated convergence.}

Then there exists a standard, regular social objective that—for any $M > 0$—locally rationalizes $R$ within the sub-space of deviations $\Delta^*_M \equiv \left\{ \Delta \in \Delta \mid ||\Delta|| \leq M \left[ \int g(z) (R(z) + z) \left( \frac{(\Delta(z))^2}{R(z)^2} + \left( \frac{\Delta'(z)}{R'(z)} \right)^2 \right) dz \right]^{\frac{1}{2}} \right\}$. (13)

**Proof.** See Appendix C.2

Considering that (DEFG) reflects only the robustness of $R$ to very “narrow” deviations in taxes, this is a surprising result. Indeed, given a fixed social objective, we know that the planner must consider the robustness of $R$ to all possible deviations. Why do we not need additional conditions in order to rule out these other second-order deviations?

The main insight is as follows: For an arbitrary planner, one must indeed verify additional second-order conditions corresponding to variations in taxes not spanned by “narrow” deviations. However, there exists a planner for whom all other deviations are undesirable. Intuitively, any non-narrow deviation has an appreciable effect on the level of income retention, instead of just its slope. While changes in the level of retention have
several effects, one of them corresponds the curvature of the planner’s re-distributional preferences: As she transfers more money to any one household, her desire to do so may increase or decrease. By taking the planner’s preferences to be sufficiently risk-averse around each household’s initial utility level, we can always make negative the second-order effects of tax variations that significantly vary the level of retention. Since \textit{DEFG} ensures all variations that do not significantly vary the level of retention have negative second-order effects, this is sufficient to ensure local optimality.

While our focus on rationalizability facilitates this clean characterization, we also provide results relevant to the approach that fixes a welfare function and solves for optimal taxes. Specifically, Lemmas 6 and 7 provide the planner’s full first- and second-order conditions.

5 Empirical test of rationalizability

We now set out to empirically evaluate the \textit{ABC} and \textit{DEFG} tests. To do so, we estimate each of the sufficient statistics that comprise them in the NBER tax return datasets, which include both a panel of household tax returns and larger repeated cross sections within each year. After evaluating the test, we check the robustness of our estimates using direct evidence that tax changes can indeed “sort” households by elasticities.

To set the stage for our estimation, recall our second-order \textit{DEFG} test for “Laffer valleys.” In the simplest case where (a) the distribution of elasticities is locally constant in $z$, (b) taxes are locally linear in $z$, and (c) labor supply preferences are additive CES, the test simplifies to

\[(1 + R'(z))\varepsilon(z) \leq (1 - R'(z))\alpha(z)\left(\varepsilon(z)^2 + \text{Var}[\varepsilon(R)|z_0 = z]\right).
\]

This expression serves two expository purposes. First, it clarifies what are our main statistics of interest: The shape of the tax schedule ($R'(z)$), the shape of the income distribution ($\alpha(z)$), income-conditional mean compensated elasticities of taxable income ($\varepsilon(z)$), and the income-conditional variance of these elasticities. Of these statistics, income-conditional variance is the most novel to our analysis and poses the greatest estimation challenges.

Second, \textit{14} facilitates simple, back-of-the-envelope calculations. In the US, top marginal taxes are roughly $R'(z) \approx 0.5$, the income distribution features a Pareto tail with shape $\alpha(z) \approx 2.5$, and existing elasticity estimates—while they vary widely—are in

\footnote{While we provide these conditions in full, they (a) require the estimation of many statistics in order to be evaluated and (b) do not make it simple to verify the second-order condition even given perfect information. Our rationalizability approach avoids both of these issues. For a discussion of (b), see the second paragraph of Section C.2.}
the vicinity of $\varepsilon(z) \approx 0.3$ [Gruber and Saez, 2002; Saez et al., 2012]. So, the second-order condition fails if and only if the variance of elasticities of top earners is $\geq 0.27$. Indeed, this number is close to the lower bound on variance that is implied by the difference of elasticity between itemizers and non-itemizers estimated by Gruber and Saez [2002]: these groups are roughly evenly sized and their mean elasticities differ by about one, implying the variance of elasticities across only the two groups is about one quarter. We will estimate that within-income variances that are significantly higher, implying that taxes are in a Laffer valley.

5.1 Data

We use the NBER panel and repeated cross-section of tax returns from 1979 to 1990. This sample period includes major tax reforms such as the Economic Recovery Tax Act 1981, which decreased marginal rates in 3 years from 1982 to 1984; the Tax Reform Act 1986, which decreased the number of brackets, and reduced the top marginal rate to 28%; the 1987 EITC expansion; and some state level tax reforms.

The data include limited demographic information, individual-specific federal and state income tax schedules and various measures of income. To construct a consistent measure of taxable income for the whole period, we closely follow Gruber and Saez [2002]. The measure we use includes wage, business and capital income, and subtracts exemptions, standard and itemized deductions. Within the panel data, we—following the procedure that Gruber and Saez [2002] use to compute medium run elasticities—compute income changes over 3-year windows. We compute marginal tax changes (at initial income) over the same windows, and we drop individual-years with changes in marital status, initial (pre-deductions) income lower than $10$ thousand dollars (in 1990 terms).

The panel sample consists of a random selection of four digit endings of social security numbers. There is purely random attrition; in some years a random subset of the sample social security numbers are excluded. The repeated cross-section sample is larger but not a panel; it over-represents higher-income individuals, which we account for with sample weights. Table 1 provides summary statistics.

While this data set may not offer the cleanest imaginable identification—tax changes are not randomly assigned to households at the individual level—we believe it is appropriate for our exercise. For one, it contains tax changes that affect people throughout the income distribution, which allows us to estimate ETIs at each income level. Second, the data includes substantial tax variation at both the state and federal level, and as such workers’ behavioral responses to these changes can be thought of as typical for real tax

\footnote{These are computed using the NBER TAXSIM program, which calculates liabilities under US Federal and State income tax laws from individual data.}
Table 1: Summary statistics for panel data and repeated cross sections (CS)

<table>
<thead>
<tr>
<th></th>
<th>Panel: Mean</th>
<th>SD</th>
<th>CS: Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taxable income (1990 dollars)</td>
<td>29,489</td>
<td>48,501</td>
<td>286,603</td>
<td>1,222,002</td>
</tr>
<tr>
<td>Single dummy</td>
<td>0.30</td>
<td>0.46</td>
<td>0.22</td>
<td>0.42</td>
</tr>
<tr>
<td>Tax rate (state+federal)</td>
<td>28.37</td>
<td>9.33</td>
<td>35.87</td>
<td>14.07</td>
</tr>
<tr>
<td>Change in taxes at initial income</td>
<td>-1.86</td>
<td>4.71</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log change in income</td>
<td>0.03</td>
<td>0.85</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of observations</td>
<td>59,199</td>
<td></td>
<td>1,380,590</td>
<td></td>
</tr>
<tr>
<td>Max # obs in a year</td>
<td>10,717 (1987)</td>
<td></td>
<td>203,448 (1979)</td>
<td></td>
</tr>
<tr>
<td>Min # obs in a year</td>
<td>4,448 (1983)</td>
<td></td>
<td>76,134 (1983)</td>
<td></td>
</tr>
</tbody>
</table>

reforms. Finally, it contains tax changes of different sizes, and one of our strategies for estimating ETI variance leverages this variation in treatment size.

5.2 Estimation of sufficient statistics

In order to evaluate our rationalizability test, we require estimates of the shape of income tax schedule, the shape of the income distribution, and several moments of households compensated, income, and super-ETIs conditional on income. Of these, the last is by far the most ambitious relative to the existing literature, which typically estimates only mean uncompensated within coarse income bins.

To make this task possible, we impose three structural assumptions:

1. Households have no income elasticity.
2. Households have CES disutility of labor supply.
3. Households respond fully to changes in taxes within three years.

These assumptions allow us to focus on estimating the income-conditional moments of medium-run compensated elasticities\(^{32}\). While these assumptions are strong, they are common in the literature and we feel that—despite them—our estimates provide a plausible assessment of our rationalizability test. Still, our results should all be interpreted with significant caution.

5.2.1 Mean elasticities

We begin by estimating mean ETIs by income level. Our empirical strategy closely mirrors that of Gruber and Saez [2002], except that—instead of estimating a single ETI—we estimate ETIs locally in the space of year-demeaned log income \(\tilde{z}\). Concretely, we

\(^{32}\)They imply that compensated elasticities are equal to uncompensated elasticities and that there households have no super-elasticities, except through changes in the curvature of the tax schedule.
estimate the following local regression, weighting observations by the distance between
\( \log z^h_t \) and \( \tilde{z} \):

\[
\Delta \log z^h_t = a(\tilde{z}) + \varepsilon(\tilde{z}) \cdot \Delta \log R'_t(z^h_{t-1}) + c(\tilde{z}) \cdot mrs^h_{t-1} + d_{t-1} + \xi^h_t
\]  

(15)

where \( \Delta \) represents time differences in \( t \) holding \( t - 1 \) fixed, so that \( \Delta \log R'_t(z^h_{t-1}) = \log R'_t(z^h_{t-1}) - \log R'_t(z^h_{t-1}) \) is the change in marginal retention at \( h \)'s initial income. The year-demeaned-log-income-specific constant \( a(\tilde{z}) \) controls for differences in typical income changes in different brackets so that our estimates are not biased by, for example, mean reversion. We also control for a marital status dummy and time fixed effects.

Methodologically, we implement this regression by first differencing out by year fixed-effects and then running local-polynomial regressions in year-demeaned-log-income space with a constant bandwidth and using the Epanechnikov kernel. We optimally select this bandwidth using a leave-one-out cross validation procedure. We compute confidence intervals by using the basic bootstrap method described in Chapter 5 of Davison and Hinkley [1997].

Under the assumption—which we maintain throughout—that, conditional on our controls, changes in marginal tax rates within a (demeaned) income level are as good as randomly assigned, this regression identifies the average ETI of households with each year-demeaned log-income \( \tilde{z} \). This is exactly the statistic required by our theory. Given the presence of year fixed effects, our identifying variation consists of (a) within-year, within-income variation in tax changes across individuals (especially those living in different states) and (b) across-year variation in the relative tax rates between different income levels. Year fixed effects control for the fact that the government may adjust taxes in anticipation of changes in aggregate economic conditions, so long as these tax changes are uniform across tax brackets.

Figure 4 shows our estimates of ETI by income level. Our estimates are consistent with—though somewhat on the lower end of—estimates in the literature. In line with

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33 For more details, see Appendix D.1.

34 Our estimation strategy differs from some earlier work, such as that of Gruber and Saez [2002], which identifies a hypothetical elasticity concept based on locally-linear taxes. Our elasticity concept is inclusive of changes in income caused by “knock-on” changes in marginal taxes as a household adjusts its income.

35 Decomposing these sources of variation helps to clarify in what cases our identifying assumptions are violated. For example, variation through (a) violates the identifying assumptions if incomes decline in a state in the same year that state has a tax change, but for unrelated—and yet not statistically independent—reasons. Variation through (b) violates the identifying assumptions if incomes at some income level decline differentially (relative to those at other income levels) in the same year that taxes change differentially across those brackets, but for unrelated—and yet not statistically independent—reasons.

36 As discussed above, we estimate ETI locally in year-demeaned-log-income space. Figure 4 converts these estimates to income space in 1990 by combining log income levels that are the same distance from average log incomes in their respective years. Appendix Figure 12 shows the original estimates in year-demeaned-log-income space.
Gruber and Saez [2002] and Kumar and Liang [2020], we find somewhat higher elasticities at the bottom and the top of the income distribution.

![Graph showing mean ETI by income level in 1990 USD, and 95% confidence bands.](image)

Figure 4: Mean ETI by income level in 1990 USD, and 95% confidence bands.

5.2.2 Variance of elasticities

Our second object of interest is the variance of ETI conditional on income level. While there is some precedent for studying differences in ETIs across groups, our approach varies in two main ways. First, the existing literature does not emphasize within-income variation and in some cases explicitly focuses on across-income variation [Eissa and Liebman, 1996, Gruber and Saez, 2002, Kopczuk, 2005, Kumar and Liang, 2020]. By contrast, our explicitly calls for within-income variance estimates. Second, the literature has focused on differences in elasticity conditional on observed characteristics. However—insofar as our test does not apply when taxes can condition on observed heterogeneity—we are particularly interested in unobserved heterogeneity (see Section 4.3.1). This focus brings with it additional identification challenges, which we discuss.

We pursue two main strategies: (a) a conditioning on observables approach based on differences across itemization status and (b) a structural estimation approach that relies on the linearity of tax responses.

Conditioning on observables: itemization status

Motivated by work that has documented higher elasticities for households who itemize deductions, we use the number of itemizations to split our sample into heavy and low itemizers [Gruber and Saez, 2002, Kopczuk, 2005]. While we could simply compare household who do and do not itemize (at all), this would be problematic in our setting
because almost all high-income households do itemize. So that we have variation in 
itemization status at each income level, we categorize households as low or high itemizers 
depending on whether they have below or above the mean level of itemizations at their 
income level. We define itemization status in year before the tax changes we consider, 
so that it does not implicitly control for changes in income.

After classifying individuals into these two groups, we estimate the local regression at 
each year-demeaned income level $\tilde{z}$:

$$
\Delta \log z_t^h = a(\tilde{z}) + \varepsilon_{L,t}(\tilde{z}) \cdot \Delta \log R_t'(z_{t-1}^h) + \delta(\tilde{z}) \cdot HI_{t-1}^h \cdot \Delta \log R_t'(z_{t-1}^h) + c(\tilde{z}) \cdot mrs_{t-1}^h + d_{t-1} + \xi_t^h
$$

where $HI_{t-1}^h$ is an indicator for high itemizers. This differs from our estimation of unconditional mean elasticities in two ways. First, we interact tax changes with a high income dummy in order to estimate the difference in elasticity by itemization status. Second, by allowing $\varepsilon_{L,t}(\tilde{z})$ to differ across years, we ensure that $\delta(\tilde{z})$ measures only within-year differences in elasticities by itemization status, despite the fact that during our sample period, a broadening of the tax base during our sample period may have both reduced elasticities and reduced itemizations [Kopczuk, 2005].

We estimate this regression locally in year-demeaned log-income space with the same methodology described in Section 5.2.1. To compute confidence intervals, we bootstrap both steps of the procedure just described, holding fixed the choices for the bandwidths, and use the basic bootstrap confidence interval.

The left panel of Figure 5 shows our estimates of differences in elasticity between heavy 
and light itemizers. Next, we combine these estimates with estimates of the shares of 
heavy-itemizers by income bracket, in order to compute an implied lower bound on the 
variance of elasticities by income brackets. Again, we compute confidence intervals by 
bootstrapping the whole procedure while holding bandwidth choices fixed. The right 
panel of Figure 5 shows the implied variance as a function of income. At high incomes, 
the lower bound on variance we estimate is already close to the level required to violate 
the back-of-the-envelope second-order condition discussed above.

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37Concretely, we compute the mean number of itemizations by income bracket ($n_{item}$) non-
parametrically by estimating the following local regression in the space of year-demeaned log income $n_{item}^h_{t-1} = a(\tilde{z}) + b(\tilde{z}) \log z_t^h + \xi_{t-1}^h$.

Here, $a(\tilde{z})$ is our actual estimate, but we include the linear term as recommended by Fan and Gijbels [1996].

38Had we not allowed $\varepsilon_{L,t}(\tilde{z})$ to depend on year, we may have picked up between-year differences related to that there may be relatively more itemizers in years where both itemizers and non-itemizers have high elasticities—in which case the difference between average elasticities of itemizers and non-itemizers across years could exceed the average difference within years—or vice-versa.

39Appendix Figure 13 show the analogous plots of Figure 5 in the space of year-demeaned log income, where we perform the actual estimation before mapping to 1990.

40We estimate these shares by the same local-polynomial approach used for our other regressions.
This approach generates a conservative but robust estimate of the variance in ETI by income level. It is conservative because it leverages only differences in elasticities across two observable groups, ignoring observed and unobserved variation within these groups. At the same time, it is robust to the concern that (due to super-elasticities or tax salience) households’ tax responses may be non-linear in the size of tax changes. Such non-linearities could cause us to mistake variances in the size of tax changes for variance in ETI at a given tax change; this is not an issue when estimating means, so long as each group faces similarly-sized tax changes.

**Structural estimation**

In order to estimate unobserved heterogeneity in elasticities, we complement the approach above with a second, more structural approach.

The key idea underlying this approach that if (a) each household’s elasticities do not depend on the size of the tax changes they face and (b) one has access to continuous variation in the size of tax changes, then all moments of the distribution of elasticities—and in particular its variance—are non-parametrically identified. Intuitively, linearity implies that as the treatment size increases, the variance of treatment responses can be identified out of the change in the dispersion of the outcomes. This is because the heterogeneity in treatment effects should progressively magnify the dispersion of outcomes as the size of the treatment increases.\(^{41}\)

While linearity is, admittedly, a strong assumption, it allows us to remain very flexible in modelling the structure of ETI heterogeneity. In particular, we assume that the distribution of elasticities has finite support, but allow this support to have many points, each taking arbitrary values. This flexibility allows us to approximate many different dis-

\(^{41}\)For a proof, see Appendix D.2
tributions, while discreteness ensures that our resulting estimates of ETI variance cannot be driven by a fat tail.

More explicitly, we assume that

$$\Delta \log z_h^t = a_g^t + \varepsilon_g^t \cdot \Delta \log R'_t(z_{t-1}^h)$$

(17)

where $g_h^t$ takes a finite number of values, and where $\Delta \log z_h^t$ and $\Delta \log R'_t(z_{t-1}^h)$ are changes in log income and change in log taxes at initial income, respectively, after partialling out by marital status, a 10-piece linear spline in log taxable income, and year dummies. We estimate this regression by running a k-means algorithm that minimizes the mean square error of (17) by assigning each $(h, t)$ pair to a group, and iteratively estimating $\{a_g, \varepsilon_g\}$ within groups.

The left panel of (6) displays our estimated distribution of ETIs. The estimated distribution has a very large mass around zero and then a thick tail, but to a maximum ETI of 16. While this number is large, we believe it is plausible as an intensive margin elasticity at low incomes or as a proxy for extensive margin labor supply or tax avoidance decisions at any income. For robustness, we also present versions of our main results in which ETIs are capped at 5.

Given assignments $\hat{\varepsilon}_h^t$ of ETIs to each individual at each date $t$ (i.e. between $t$ and $t + 1$), we compute the variance in ETIs by income level with local polynomial regressions.

In order to ensure that we capture only within-year variation in elasticities, we estimate the following regression locally in the space of year-demeaned log incomes $\tilde{z}$:

$$\left(\tilde{\varepsilon}_h^t - \hat{m}_t(\tilde{z})\right)^2 = a(\tilde{z}) + b(\tilde{z}) \log z_h^t + \xi_h^t$$

(18)

where $\tilde{z}_h^t$ is year-demeaned log income, and $\hat{m}_t(\tilde{z}_h^t)$ is the year and income specific mean of elasticities, also estimated using first order local polynomials.

One important concern about this procedure is that it may be prone to small sample bias, as we estimate a large number of parameters. To address potential small sample bias and to obtain confidence intervals for our estimates of variance, we bootstrap the entire procedure, holding fixed the number of groups $k$ and all bandwidths, and then subtract from our point estimates the bootstrap estimator of the bias.

The right panel of Figure 6 shows our structural estimates of income-conditional variance in ETI. Notably, we estimate significantly higher variances than with the lower

---

42 The k-means algorithm dates back to Sebestyen [1962] and MacQueen et al. [1967], and has recently been applied by Bonhomme and Manresa [2015], Bonhomme et al. [2017], Lewis et al. [2019].

43 We do this while fixing the number of groups $k$, which we then select with a Bayesian information criterion subject to a maximum possible number of groups $k_{max} = 100$ as in Bonhomme and Manresa [2015].

44 We follow an analogous procedure to compute the third moment of ETIs.

45 Here again we have combined incomes across years that share a common year-demeaned log income.
bound approach of the prior section. Appendix Figure 14 shows our structural estimates when ETIs are capped at 5.

Figure 6: Left panel: Distribution of ETIs identified by k-means procedure. Right panel: Implied variance in elasticities by income level. In gray: 95% confidence bands.

5.2.3 Estimation of tax schedule, income density

Our rationalizability test also relies on estimates of the shape of the tax schedule and the income distribution. We estimate these moments in the NBER cross sectional files, which contain more observations and so allow for more precise estimation.

To compute $\alpha(z)$ in each year, we estimate a smooth functional form for the CDF of taxable income $G_t(z)$ in each year $t$ with local polynomials regressions to third degree in log taxable income. Here we follow Fan and Gijbels [1996], who recommend including terms up to one order above the derivative of interest. We then translate our estimated coefficients for the first and second derivatives of $G_t(z)$ in log income into an estimate of $\alpha_t(z)$. We obtain confidence intervals by bootstrapping the whole procedure while holding the estimated optimal bandwidth constant. Our estimates are broadly consistent across years and similar to other recent work such as those of Hendren [2020]. The left panel of Figure 7 shows our estimates in 1990.

We take a similar approach to estimate marginal tax rates and each of its derivatives required by our test. In these cases we estimate local-polynomial regressions of order $d$, where $d$ is the nearest odd integer greater than the derivative of interest (in log income space). Again, we bootstrap confidence intervals holding optimal bandwidths fixed. The right panel of Figure 7 shows our estimates in 1990.$^47$

$^46$Namely our estimate of $\alpha_t(z)$ is equal to the negative of our local estimate of $\frac{d^2G_t(z)}{d \log z^2}$ divided by our local estimate of $\frac{dG_t(z)}{d \log z}$.

$^47$We show all years together in Appendix Figure 16 to 26.
By construction, our estimates of the tax schedule do not feature any kinks. This reflects that we run local regressions with a finite bandwidth, which smooths out the schedule. To the extent that kinks affect our rationalizability test, we therefore ignore their effects. However, we think that a smoothed tax schedule may be a realistic interpretation of the way that households perceive a kinked tax schedule, or a proxy for their inability to perfectly adjust their incomes, and so do not dwell on this issue [Rees-Jones and Taubinsky, 2020].

5.3 Evaluation of test

Having computed each of the elements in the \((ABC)\) and \((DEFG)\) formulas, we can evaluate whether the tax schedule is rationalizable.

5.3.1 First-order test

We start with the first-order test \((ABC)\), which recall assesses whether taxes are above the top of the Laffer curve at each income level \(z\). Concretely, we compute—for each year \(t\) in our data—the statistic

\[
\hat{ABC}_t(z) = -1 + \frac{1 - R'_t(z)}{R'_t(z)} \left( \alpha_t(z) - \frac{d \log \varepsilon(z)}{d \log z} \left( \frac{1 - R'_t(z)}{R'_t(z)} \right) - \frac{d \log \varepsilon(z)}{d \log z} \right) \varepsilon(z) \tag{19}
\]

where recall \(R_0\) and \(\alpha_t\) are estimated separately in each year. We compute confidence intervals for \(\hat{ABC}_t(z)\) by combining bootstrap replications from the two different data.

\(^{48}\)To ensure our test is robust to this treatment of kinks, we also estimate our second-order test with an alternative methodology that excludes the term proportional to tax curvature. This has negligible effects, see Appendix Figure \(^{39}\).
sets we use—the panel and the cross-sectional file for year $t$—assuming that observations in separate datasets are drawn independently from each other.

Our estimates of $\hat{ABC}_t(z)$ are broadly consistent across years and consistently negative, implying that taxes are below the top of the Laffer curve. The left panel of Figure 8 shows this for one representative year; Appendix Figures 27 to 37 show our estimates for all years.

5.3.2 Second-order test

We now evaluate our new, second-order test ($\hat{DEFG}$), which recall assesses whether taxes are in a Laffer valley at each income level $z$. Concretely, we compute—for each year $t$ in our data—the statistic

$$\hat{DEFG}_t(z) = - (1 + R_t'(z)) \varepsilon(z) + (1 - R_t'(z)) \left[ \alpha_t(z) \varepsilon^2(z) - \frac{d\varepsilon^2(z)}{d \log z} + \frac{d^2 \log R_t'(z)}{d \log z^2} \varepsilon^3(z) \right]$$

(20)

where the elasticities are derived from our structural estimates. We compute confidence intervals in the same way described for $\hat{ABC}_t(z)$.

Our estimates of $\hat{DEFG}_t(z)$ are broadly consistent across years. Strikingly, they fail at incomes above around 90,000 in 1990 USD. The right panel of Figure 8 shows this for one representative year; Appendix Figures 27 to 37 show our estimates for all years. In Appendix Figures 38 to 40, we compare $\hat{DEFG}_t(z)$ with and without including the last two terms of (20); this has almost no effect, implying that our estimates are not driven by the third moment of elasticities or steep changes in the second moment.

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49 This is equal to the RHS of (DEFG) when, as we have assumed, income effects are zero and labor supply preferences are additive-CES; see Appendix A.4.1 for details regarding the super-elasticity term.
An alternative way to visualize \((\text{DEFG})\)—as well as to compare the implications of our different estimates of variance—is to reframe the second-order test in terms of the minimum level of variance at which it fails. We can then compare this level with our estimates of variance, at each income level. Figure 9 illustrates this comparison\(^{51}\). Blue and orange shaded regions represent the levels of ETI variance at which the \((\text{DEFG})\) test passes and fails, respectively, at each income. On top of this background, we superimpose our estimates of variance based on the two strategies in Section \((5.2.2)\). The test easily fails under our structural estimates of variance, while our lower bounds on variance based on itemization status come about half of the way to violating the test. Not shown below (see Appendix Figure \([14]\)) are our structural estimates of variance when elasticities are capped at 5; these violate the test on point estimates but cannot reject that the test passes.

Figure 9: 1990. Background: ETI variance consistent (blue) and inconsistent (orange) with \((\text{DEFG})\). Foreground: Estimates of ETI variance. Lighter colors and dashed lines colors are 95% confidence bands.

### 5.4 Inspecting the mechanism

A number of potential confounding factors could prevent the heterogeneity in elasticities we have estimate from translating into the type of elasticity-sorting that our theory predicts. For example, perhaps our estimates mistake differences across households in the

\(^{51}\)It does so under the assumptions that, at each \(z\), (a) the tax schedule is locally linear and (b) the variance of ETIs is locally constant. This allows the test to be framed as simple “variance test” without considering the other roles played by second and third moments of ETIs. Figure 39 shows that this simplification is without significant loss.
timing of tax responses for differences in the size of tax responses.

Another possibility is that super-elasticities—which we do not estimate—could work to attenuate the level of elasticity sorting, similar to the discussion in Section 4.3.2. Alternatively, heterogeneity in elasticities across income levels may reflect differences in the institutional arrangements of employment in different jobs, which may not respond to tax changes.

To address these concerns, we now attempt to directly validate the mechanism that underlies our theory—that changes in taxes cause changes in the average elasticity within each income level. The main idea behind our strategy is as follows: In regions of the tax schedule where the income density is strongly decreasing \((x(z) > 0)\), a locally flat increase in marginal retention should increase the average elasticity conditional on income. This is because—when there is significant heterogeneity in elasticities and when all elasticity groups have proportional densities—the high elasticity types that end up at each income level \(z\) come from much lower incomes, where the density is higher. This pattern flips where \(x(z) < 0\), so that an increase in marginal retention should lower the average elasticity at \(z\).

More concretely, one can show that for small tax changes,

\[
\Delta \varepsilon(z) \approx x(z) \text{Var}[\varepsilon^h(R)|z_h^0 = z] \Delta R'(z).
\]

In other words, there is a positive interaction between the ETI \(\varepsilon(z)\) at a given income level and the product of the Pareto shape \(x(z)\) and the level of marginal retention \(R'(z)\), and the size of this interaction term is equal to the local variance of ETIs. In principle, we could therefore identify ETI variance at each income level by including in the mean ETI regression \([15]\) an interaction between tax changes and the product of \(x(z) \cdot R'(z)\). In practice, this approach has two limitations, and these limitations motivate our actual strategy. First, it may be under-powered, so we pool across income levels. Second, one source of variation in \(R'(z)\)—tax differences across states (rather than years)—may be correlated with elasticity differences for reasons other than our mechanism and therefore bias this regression. Namely, one should expect that states with more income-elastic or cross-state-migration-elastic populations (at any income \(z\)) respond by levying lower marginal taxes. We therefore focus on the component of variation in taxes that comes solely from time variation.

Our preferred specification is as follows:

\[
\Delta \log (z^h_t) = \varepsilon \cdot \Delta \log R'(z^h_{t-1})
+ \gamma \cdot \Delta \log R'(z^h_{t-1}) \cdot (\log R_{t-1}(z^h_{t-1}) - \overline{\log R(z^h_{t-1})) \cdot \alpha_{t-1}(z^h_{t-1})}
+ c \cdot mrs_{t-1} + d_{t-1} + f(z^h_{t-1}) + \xi^h_t
\]

\[52\] Our conditioning-on-observables estimates are robust to this concern, but not our structural estimates are not.
where $\log \overline{R'}(z^h_t)\) is the average log marginal retention rate at $z^h_t$ across years, and where $f(z^h_{t-1})$ is a control for the level of income (either a single linear term or ten piece splines). As discussed above, one may interpret the regression coefficient on this interaction term as a measure of the average variance in ETIs across tax brackets. Secondarily, we also estimate (22) using only cross-state variation—in which case the cross-state within-year average shape $\alpha_{t-1}(z^h_{t-1})$ and average marginal taxes $\log \overline{R'}_{t-1}(z^h_{t-1})$ are replaced by the analogous within-state across-year average shape and average marginal taxes for which we estimate the full cross section of $\alpha$s and $\log \overline{R}'s$.

Table 2 presents our results. In most cases we estimate positive but small and statistically insignificant implied variances, which we interpret as moderately supportive of our theory. The small size of our estimates using year $\times$ state variation is likely explained by attenuation bias stemming from noise in our estimates of local $\alpha$s.

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Notes. Estimates for means and variances of elasticities using equation [22]. Bootstrapped standard errors in parentheses. The first two columns report results where the variance of elasticities is inferred from variation in elasticities across years. In the third and fourth column, it is inferred from variation across states, and in the fifth and sixth it is inferred from variation across years and states.

Finally, we extract an implicit measure of variances by income bracket by running the regression above non-parametrically, allowing $\gamma(z), \epsilon(z)$ and $c(z)$ to depend on demeaned income. Figure 10 shows that, in the year-variation-only case, we estimate implicit variances similar to those in Section 5.2.2. In the other specifications, our estimates are very noisy and close to zero; see Appendix Figure 15. As we believe the year-only variation is better identified, we conclude that this exercise lends moderate support to our theory overall.

6 Quantitative application

So far, we have presented evidence that the taxes in our sample can—from the perspective of any planner in a broad class—be improved. But how large are these potential welfare gains, and what sort of tax changes achieve them? In this section, we speak to these questions using a simple calibrated model.
Figure 10: Income-conditional variance implied by elasticity differences across low- and high-tax years in 1990, 95% confidence bands.

Model: Motivated by the fact that our efficiency test fails only at high incomes, we remain agnostic to the behavior of households with incomes below $100,000 and to the planner’s preferences regarding their welfare. For households with incomes above $100,000, we impose both positive and normative assumptions. Positively, we assume the following: First, consistent with the year 1990, top incomes are distributed as Pareto ($\alpha_{top} = 2.5$), are subject to an initial top marginal tax rate of $\tau_{top} = 35\%$. Second, each top-earning household $h$ has a constant compensated elasticity of labor supply $\beta_h$ and no income elasticity. Third, $\beta_h$ is distributed independently of productivity according to a Gamma distribution with mean 0.3 and variance 1.2\(^{53}\).

Normatively, we must take a stance on how the planner values changes in household welfare relative to government revenue in order to quantify the gains from tax reform. To do so, we assume the following: First, the planner places a common and constant value $\lambda_{top}$ on transfers to each top-earner, relative to government revenue\(^{54}\). Second, in setting the initial tax schedule, the planner has “followed the literature” by checking the first- but not the second-order condition of the tax schedule. Together with our positive assumptions, this allows us to back out the welfare weight:

$$\lambda_{top} = 1 - \frac{1}{\frac{R'_{top}}{\alpha_{top}} \epsilon_{top}} \approx 0.6.$$  \quad (23)

\(^{53}\)We view this as a sensible approximation of our empirical estimates of elasticity heterogeneity.

\(^{54}\)Here, we implicitly assume that the planner does not face a hard budget constraint, but instead has a constant marginal value of public funds.
Simple tax reforms: To motivate the particular tax changes we consider, we return to the planner’s second-order condition. In our specialized model, this takes a particularly uncomplicated form for any tax change $\Delta$ that only affects top earners:

$$
\frac{d^2}{d\epsilon^2} \text{Welfare}(R + \epsilon \Delta) = \int \left[-(1 + R_{\text{top}})\epsilon_{\text{top}} + (1 - R_{\text{top}})\alpha_{\text{top}}\epsilon_{\text{top}}^2 \right] \left(\frac{\Delta'(z)}{R_{\text{top}}}\right)^2 zg(z) dz.
$$

One may easily verify that the “LHS of (DEFG)” term is positive under our positive assumptions, implying that the planner’s second-order condition fails, i.e. taxes are in the Laffer valley.

This expression says that an change in taxes on top earners improves welfare to the extent it changes the marginal taxes they face. In particular, there is no need to restrict to very “narrow” tax changes—as was done in the main text—in order to exploit the failure of the second-order condition; narrow tax changes are just a technical device used to handle income effects and non-constant welfare weights, both of which are absent here. Even simple tax changes, such as raising or lowering marginal taxes on all top earners, can “sort and extort,” as illustrated in the motivating example of Section 2.

This in mind, we consider two very simple tax reforms: A 20 percentage point increase and a 20 percentage point decrease in the top marginal tax rate.

Welfare gains and discussion:

Consistent with the observation that taxes are in a Laffer valley, we document welfare gains from either raising or lowering taxes. Concretely, the planner’s gains from either raising or lowering top taxes by 20pp are equivalent to the social value of transferring a little more than $3000 to each household in the top bracket or $\lambda_{\text{top}} \cdot $3000 $\approx$ $2000 to the planner for each household in the top bracket.

The estimate above is a lower bound on welfare gains for several reasons. First, we take all taxes below $100,000 as given, thus ignoring the potential gains from simultaneously changing them. Second, we restrict taxes above $100,000 to be linear, ignoring the additional gains from using non-linear tax changes. Third, we limit ourselves to 20 percentage point changes in the top tax rate, ignoring the gains from using larger tax changes. Fourth—and somewhat more subtly—we ignore any behavioral effects on households with initial incomes outside of the top bracket. This results in an underestimate since the only case in which such households respond to top rate changes is when they jump into the top bracket, which has positive fiscal externalities.

One topic not addressed above is the shape of the optimal tax reform, and in particular whether it is similar to the simple reforms we consider. For instance, one may worry that a schedule with many “squiggles” would be more effective in sorting households.

55This follows from Lemmas 6 and 7, see Appendix C.1
with different elasticities and therefore optimal. While we do not answer this question in general, Appendix B.3 presents a suggestive exercise in which, starting from a suboptimal tax schedule, we update taxes in the direction suggested by the planner’s first-order condition until converging to a new local optimum. In a simple case with two elasticity types, we find that taxes converge to a “two-part” schedule that smoothly transitions between a high tax rate used on low incomes and a low tax rate used on high incomes—rather than a “squiggly” schedule.

7 Discussion

7.1 Relation to inverse optimality literature

Overall, we interpret the empirical violation of our rationalizability test as new and surprising observation about the (in)efficiency of US income taxes. The existing literature—typified by the first-order (ABC) test—has tended to interpret the US income tax schedule as Pareto efficient and reflective of a particular set of social preferences that places somewhat greater weight on transfers to lower-income households [Bargain et al., 2014]. Our interpretation differs in two ways: First, the violation of our second-order test implies that taxes are, for any social planner in the broad class we consider, inefficient. Second, not only are taxes inefficient, but also—because they are un-rationalizable,—it is misguided to interpret taxes as reflecting any set of social preferences. In this sense, our findings are critical of the “inverse optimum” literature which attempts to infer a planner’s distributional preferences from the tax schedule.

At the same time, our results are consistent with a more recent interpretation of the inverse optimum approach which shows that first-order-condition-implied “as if” welfare weights can be used to value the distributional impacts of policies in a welfare-function-independent way [Hendren, 2020, Hendren and Sprung-Keyser, 2020]. Specifically, Hendren [2020] shows that if the weighted incidence of a policy change is positive, then—if it is accompanied by an appropriate change in income taxes—it can be used to create a Pareto improvement.\footnote{Although this result is derived in a one-dimensional model, a version of it carries over to our setting if policy changes have homogeneous impacts on households within each income level.} While this result does not rely on our second-order condition, its interpretation is somewhat different when the second-order condition fails. Namely, the compensatory adjustments do lead to Pareto improvements, but also operate strictly within the frontier, since any planner could improve welfare through further tax changes.

7.2 Robustness of ETI variance estimates

We have presented a variety of estimates for variances of elasticities, some more conservative and others requiring stronger assumptions.
Perhaps the most robust are the implicit estimates of variance presented in Section 5.4 because they directly test the sorting method that underlies our theory. As we have discussed, these estimates are robust to concerns that super-elasticities may undo the sorting effects of ETI variance or that heterogeneity in elasticities across income levels reflect differences in the institutional arrangements that do not respond to tax changes. However, these estimates are also the weakest empirically. They may also be confounded by other changes in the tax system over the decade. In particular, elasticities may have changed because of the broadening of the base Kopczuk [2005] that occurred simultaneous to the lowering of marginal taxes. In this case our estimates should be biased downwards at the top (where larger variances should translated into a higher increases in elasticities), and upwards at the bottom (where larger variances should translated into lower increases in elasticities).

Our lower-bound estimates of variance based on itemization status are also somewhat robust, and to different concerns. For one, they are not subject to the criticism above about base broadening. Also, as we discuss in Section 5.2.2 they do not risk mistaking non-linearities in tax responses—i.e. heterogeneity in ETIs across sizes of tax changes—with heterogeneity in ETIs for tax changes of a given size.

By contrast, our structural estimates—which recall rely on the linearity of household tax responses—are much less robust. Of course, they also have the advantage of flexibly estimating unobserved heterogeneity. One potential concern to which they are robust, and which we have no yet discussed, is that household preferences may follow a dynamic process through time, so that the same individual who is very elastic or very productive in one year is less so during the next. So long as the distribution of types in the population is constant, this reshuffling has no effect whatsoever on our results.

Taken together, our estimates present a consistent and robust picture of the variance of ETIs by income level. In particular, our estimates suggest that the [DEFG] is very likely to fail for high incomes, even if a few of the possible identification issues discussed above are active.

A final potential concern—and one which may pose a more serious issue—is that our theory only models intensive labor supply decisions, whereas actual ETIs may involve extensive margin decisions. For example, households may join or leave the labor force, find a second job, migrate between states, or hire a tax accountant. The labor force participation margin is not a major concern for us, since we exclude households with zero income in either the pre- or post-year of our estimation. Insofar as they can be done more than once, other extensive margin decisions may be more amenable to our mechanism: The planner may, for example, sort households into a certain tax bracket by inducing them to take a second job and then tax them in anticipation of their starting a third job. However, these issues are somewhat outside of our theory in its present form; we view them as an interesting area for future work.
8 Conclusion

We take a second-order approach to the classical non-linear income taxation problem. Far from a technical detail, the second-order condition introduces a new qualitative idea for income taxation: Taxes must not only be below the top of the Laffer curve, but also must not lie in a “Laffer valley”.

Our theoretical results shed light on the important relationship between the Laffer valley and household heterogeneity. As our discussion emphasizes, heterogeneity in household elasticities within income levels provides a tax reform motive for planners who are constrained to use a single income tax schedule. By changing taxes once, the planner can (in an imperfect way) sort high- and low-elasticity households into different parts of the income distribution; by changing taxes again, she can exploit this separation, as if she had access to elasticity-dependent taxes. We capture this insight in a simple test for the local rationalizability of the tax schedule in terms of locally estimable sufficient statistics.

Our empirical results take this novel test to the data in order to assess whether actual US tax schedules from 1979 to 1990 were rationalizable by any planner. We extend the approaches of existing empirical work to estimate ETIs by income level and to estimate not only ETI means but also ETI variance—a key statistic for our theory. Strikingly, our estimates reject the rationalizability of the tax schedule in every year of our sample. Said differently, any planner in the class we consider would prefer a different tax schedule; there is a free lunch available through tax reform. A conservative quantification exercise suggests that either raising or lowering top taxes by 20 percentage points results in yearly welfare gains equivalent to approximately $3000 per top earner.

References


Lennard Bakker. Uniform convergence and differentiation. URL [https://math.byu.edu/~bakker/M341/Lectures/Lec29.pdf](https://math.byu.edu/~bakker/M341/Lectures/Lec29.pdf)


Laurence Jacquet and Etienne Lehmann. Optimal income taxation when skills and behavioral elasticities are heterogeneous. 2015.


Appendices

A  Formalized statements from the main text

Below, we provide formal statements of several definitions and assumptions stated loosely in the main text. These are organized into statements about taxes and tax changes, conditions on household and aggregate labor supply, social objective definitions, and supporting concepts for the rationalizability test. Within these subsections, we also provide basic technical Lemmas that illustrate the roles of several of the assumptions and provide a foundation used in the proofs of our main results.
A.1 Taxes and tax deviations

We begin with a basic regularity condition on taxes.

**Assumption 1.** $R$ is continuous on $\mathbb{R}_{>0}$ and three-times continuously differentiable on $\mathbb{R}_{>0}$, and there exists $B^{R} > 0$ such that for all $z \in \mathbb{R}_{>0}$,

$$
\frac{dR(z)}{d\log z} \leq B^{R}|R(z)|, \quad \text{and} \quad \frac{dR'(z)}{d\log z}, \quad \frac{d^2R'(z)}{d\log z^2} \leq B^{R}|R'(z)|. \quad (25)
$$

Next, we define a space of feasible tax changes $\Delta$ by

$$
\Delta = \left\{ \Delta : \mathbb{R}_{>0} \to \mathbb{R} \mid \Delta \text{ continuous, } \Delta \text{ three-times continuously differentiable on } \mathbb{R}_{>0}, \text{ and } \exists B \in \mathbb{R} : \forall z \in \mathbb{R}_{>0}, |\Delta(z)| \leq B|R(z)| \text{ and } |\Delta'(z)|, \left| \frac{d\Delta'(z)}{d\log z} \right|, \left| \frac{d^2\Delta'(z)}{d\log z^2} \right| \leq B|R'(z)| \right\}. \quad (26)
$$

This space is well-defined under Assumption 1. Note that the function $||\cdot||$ defined in (7) is well-defined on $\Delta$. The following Lemma establishes that not only is $||\cdot||$ a norm on $\Delta$, but also $(\Delta, ||\cdot||)$ is a Banach space. We later leverage this fact in order to apply existing results on optimization in Banach spaces.

**Lemma 1.** $(\Delta, ||\cdot||)$ is a Banach space.

**Proof.** See Appendix E.1. \hfill \square

Throughout the paper, we will consider many functions of the form $f : R + \Delta \to \mathbb{R}$, where $R + \Delta \equiv \{ R + \Delta \mid \Delta \in \Delta \}$. As any such function may alternatively be understood as a function $\tilde{f}(\Delta) = f(R + \Delta)$ on $\Delta$, we will WLOG refer to such functions $f$ as being Frechet in $\Delta$ when the corresponding $\tilde{f}$ is, and with derivatives equal to those of $\tilde{f}$. Notationally, we denote the Frechet derivative of any function $f : R + \Delta \to \mathbb{R}$ evaluated at a point $\tilde{R}$ by $Df(\tilde{R})$, if it exists. For any $\Delta \in \Delta$, we let $D_{\Delta}f(\tilde{R})$ denote $Df(\tilde{R})(\Delta)$, i.e. the first Frechet derivative of $f$ at $\tilde{R}$ in direction $\Delta$ with magnitude $||\Delta||$ (and similarly for higher derivatives).

A.2 Labor supply regularity conditions

Our first assumption on labor supply is a basic regularity condition on household preferences, satisfied by typical functional forms used in the literature.

**Assumption 2.** Household utility is given by a function $u : \mathcal{H} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R} \cup \{-\infty\}$. On the restricted domain $\mathcal{H} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, utility $u^{h}(c, z)$ is finite, has three continuous derivatives in $(c, z)$ which are measurable in $(h, c, z)$ and satisfy $u^{h}_{c}(c, z) > 0$ and $u^{h}_{z}(c, z) < 0$.

Our next assumption is of more qualitative importance. It is a set of three conditions which together guarantee that—locally to $R$—each household supplies labor purely on

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57 Throughout the paper, we interpret $\mathbb{R}$ as a measure space with respect to the linear Borel $\sigma$-algebra and the Lebesgue measure. The only spaces other than (subsets of) $\mathbb{R}$ and $\mathcal{H}$ on which we refer to measurability are products thereof; we interpret all such spaces as $\sigma$-algebras with respect to the canonical product $\sigma$-algebra.
the intensive margin. In words, they are as follows: First, each household \( h \)'s problem has a unique solution at \( \bar{R} \). Second, \( h \)'s labor supply preferences—given the tax schedule—have some positive level of concavity locally, i.e., within a ball of radius \( \epsilon^h \) around \( \log z^h_0 \). Third \( h \) has a strong enough preference for supplying labor at \( z^h_0 \) relative to any level outside the local neighborhood of concavity. Importantly for later results, the level of concavity and the relative consumption preference are both uniform across households. Intuitively, these conditions hold when households have sufficiently concave preferences relative to the curvature of the tax schedule.

**Assumption 3.** There exist \((\epsilon^h)_{h \in \mathcal{H}}, \eta, \bar{c} > 0\) with \( \epsilon^h \) \( h \)-measurable, such that for all \( h \in \mathcal{H} \)

- The problem \( \max_{z \in \mathbb{R}_{\geq 0}} u^h(R(z), z) \) has a unique, strictly positive, and \( h \)-measurable solution \( z^h_0 \) at \( \bar{R} = R \),
- There exists a function \( v^h(\bar{z}) : B_{r^h}(\log z^h_0) \to \mathbb{R} \) such that, for all \( z \in e^{B_{r^h}(\log z^h_0)} \),
  \[
  u^h(R(z)e^{v^h(\log z)}, z) = u^h(\epsilon^h_0, z^h_0) \quad \text{and} \quad v^h(\log z) \geq \eta \]
  \[
  (27)
  \]
  - and for all \( z \not\in e^{B_{r^h}(\log z^h_0)} \),
  \[
  u^h(\epsilon^hR(z), z) \leq u^h(\epsilon^{-\bar{c}}_0, z^h_0). \]
  \[
  (28)
  
  The following Lemma establishes that, indeed, the assumptions stated so far guarantee each household’s labor supply—locally to the initial tax schedule—is well-defined and unique, purely intensive, and well-behaved.

**Lemma 2.** There exists \( \delta > 0 \) such that \( z^h(\bar{R}) \) and \( \epsilon^h(\bar{R}) \) are well-defined, \( \mathcal{H} \)-measurable, and strictly positive on \( \mathcal{H} \times (\bar{R} + B_\delta(0)) \), and on this domain have two continuous and \( \mathcal{H} \)-measurable Frechet derivatives in \( \bar{R} \).

**Proof.** See Appendix E.2

In order to state the next assumption, we first introduce the concepts of labor supply compensated and income elasticities and “super-elasticities”. The compensated (income) elasticity of labor supply for a household describe how labor supply changes due to changes in local marginal (the level of) taxes fixing the local level of (marginal) taxes:

\[
\eta^h(\bar{R}) = \frac{1}{R(z^h(\bar{R}))} \frac{d \log z^h(\bar{R}^h(\cdot; \epsilon_0, \epsilon_1))}{d \epsilon_0} \bigg|_{\epsilon_0=0, \epsilon_1=0}, \quad 
\epsilon^h(\bar{R}) = \frac{1}{R'(z^h(\bar{R}))} \frac{d \log z^h(\bar{R}^h(\cdot; \epsilon_0, \epsilon_1))}{d \epsilon_1} \bigg|_{\epsilon_0=0, \epsilon_1=0}
\]

where \( \bar{R}^h(z; \epsilon_0, \epsilon_1) = \bar{R}(z) + \epsilon_0 + (z - \bar{z}^h(\bar{R}))\epsilon_1 \). The super-elasticities—denoted by \( \eta^h_{+0}(\bar{R}) \), \( \eta^h_{+1}(\bar{R}) \), \( z^h_{+0}(\bar{R}) \), and \( z^h_{+1}(\bar{R}) \)—are defined as the *indirect* change in these elasticities caused by the change in the curvature of preferences that (for non-CES preferences) results when labor supply and income respond to tax changes. The super-elasticities denoted with “+0” correspond to changes in elasticities induced by changes in the level of taxes, whereas those denoted with “+1” correspond to changes induced by changes in the slope of taxes.

\[\text{[58]}\text{Here, } e^{B_{r^h}(\log z^h_0)} \equiv \{ z \in \mathbb{R}_{>0} | \log z \in B_{r^h}(\log z^h_0) \}. \text{ Also note that any such function } v^h \text{ is unique since consumption utility is strictly increasing. Also } v^h(z) \text{ is twice continuously differentiable since } u^h \text{ and } R \text{ are and } u^h > 0, \text{ by the implicit function theorem.}\]

\[\text{[59]}\text{We present complete and formal definitions of elasticities and super-elasticities in Appendix E.2.}\]
Assumption 4. There exists \( \delta, M > 0 \) such that:

- At \( R \), pre- and post-tax income, \( z^h_0 \) and \( c^h_0 \), are \( \mathcal{H} \)-integrable.
- If \( \bar{R} \in R + B_\delta(0) \), then for all \( h \in \mathcal{H} \)
  \[
  |\eta^h(\bar{R})|, |\varepsilon^h(\bar{R})|, |\varepsilon^h(\bar{R})|, |\eta^h_{+1}(\bar{R})|, |\varepsilon^h_{+1}(\bar{R})|, |\varepsilon^h_{+0}(\bar{R})| \leq M. \tag{30}
  \]

The following Lemma establishes that, under these integrability assumptions and elasticity bounds, not only individual but also aggregate labor supply and consumption are defined and well-behaved locally to \( R \).

Lemma 3. There exists \( \delta > 0 \) such that \( z^h(\bar{R}), c^h(\bar{R}) \), and their first two Frechet derivatives in \( \bar{R} \) are bounded (as linear maps) across all \( \bar{R} \in R + B_\delta(0) \) by linear combinations of \( z^h_0 \) and \( c^h_0 \) and, in particular, are \( \mathcal{H} \)-integrable. \footnote{60}

Proof. See Appendix \[E.3]\mark.

Next, we impose conditions that make it possible to express aggregate tax revenue in terms of an integral over income levels, which allows us to integrate by parts.

Assumption 5. The distribution of initial pre-tax income, \( z^h_0 \), admits a twice-continuously differentiable density \( g \) on \( \mathbb{R}_{\geq 0} \); and for each of the following elasticity variables \( x^h_n \), a conditional expectation \footnote{61}\footnote{62}\footnote{63} \( x(z) \equiv \mathbb{E}[x^h_n|z^h_0 = z] \) is \( n \)-times continuously differentiable on \( \text{supp} \, g \). This holds for \( x^h_2 = \varepsilon^h(R), \eta^h(R), \varepsilon^h(R), \eta^h(R)^2, \varepsilon^h(R)^2, \eta^h(R)^2 \varepsilon^h(R), \eta^h(R)\varepsilon^h(R)^2, \eta^h_{+1}(R) ; \) and \( x^h_0 = \eta^h(R)^3, \varepsilon^h(R)^3, \eta^h_{+2}(R), \varepsilon^h_{+0}(R), \varepsilon^h_{+1}(R) \).

Theorem 2, but not Theorem 1, relies on the following, additional regularity condition. It allows us to integrate by parts not only in a local region of the income schedule but over all incomes from zero to infinity. \footnote{64}

Assumption 6. The following, additional regularity conditions hold:

1. \( \lim_{z \rightarrow L} z^2 g(z) = 0 \) and \( \lim_{z \rightarrow L} z R(z) g(z) = 0 \) for \( L = 0, \infty \) \footnote{65}
2. For each of the following variables $x_h$ and for any $\epsilon > 0$, a conditional expectation $\mathbb{E}_2 \left[ x_n^h \mid \epsilon \right] \equiv \mathbb{E}_2 \left[ z_h^h(R) \mid \epsilon \right]$ is $n$-times continuously differentiable on $\text{supp} \ g$. This holds for $x^h_1 = \eta^h(R)$ and $x^h_0 = \epsilon^h(R)$, $\frac{\eta^h(R)^2}{\epsilon^h(R)}$.

3. Each of the following are bounded in magnitude across all $z \in \text{supp} \ g$:

$$
\left( \frac{d \log R(z)}{d \log z} \right)^{-1}, \quad \alpha(z), \quad \phi(z), \quad \phi'(z), \quad \phi''(z), \quad \phi'''(z), \quad \phi''''(z), \quad \phi'''''(z), \quad \phi''''''(z), \quad \phi'''''(z), \quad \phi''''''(z)
$$

(A.3) Social objective definitions

The basic structure imposed by the definition of a standard social objective is enough to guarantee that each individual’s contribution to aggregate welfare is locally well-behaved.

Lemma 4. Suppose that $((w_h)_{h \in H}, G)$ is a standard social objective. Then there exists $\delta > 0$ such that $w_h \circ V_h(R)$ is finite and $\mathcal{H}$-measurable on $\mathcal{H} \times (R + B_\delta(0))$, and on this domain has two continuous and $\mathcal{H}$-measurable Frechet derivatives in $\tilde{R}$.

Proof. See Appendix E.4 \hfill \Box

Toward studying aggregate welfare, we now introduce additional structure to the problem of a planner.

Definition 3. A standard social objective $((w_h)_{h \in H}, G)$ is regular if the following hold:

1. For some $\delta > 0$ and integrable functions $b_0, b_1, b_2 : \mathcal{H} \rightarrow \mathbb{R}$ such that for all $\tilde{R} \in R + B_\delta(0)$,

$$
\left| (w_h \circ u_h)(\tilde{c}^h(\tilde{R}), z_h^h(\tilde{R})) \right| \leq b_0(h),
$$

$$
\left| \tilde{c}^h(\tilde{R})(w_h \circ u_h)(\tilde{c}^h(\tilde{R}), z_h^h(\tilde{R})) \right| \leq b_1(h),
$$

and

$$
\left| \tilde{c}^h(\tilde{R})^2(w_h \circ u_h)(\tilde{c}^h(\tilde{R}), z_h^h(\tilde{R})) \right| \leq b_2(h).
$$

2. For each of the following variables $x_n^h$, a conditional expectation $\mathbb{E}_3 \left[ z_n^h \mid x_n^h \right]$ is $n$-times continuously differentiable on $\text{supp} \ g$. This holds for $x^h_0 = \lambda^h(R) \equiv (w_h \circ u_h)_c(c_0^h, z_0^h)$, $\lambda^h(R) \equiv \lambda^h(R)$, $\lambda^h(R) \frac{\tilde{c}^h(\tilde{R})}{\epsilon^h(\tilde{R})}$, $(\lambda \gamma)^h(R) \equiv (w_h \circ u_h)_c(c_0^h, z_0^h)c_0^h$, and $x^h_1 = \lambda^h(R)\eta^h(R)$.

---

Such a conditional expectation function exists since $z_0^h$ is measurable (by Assumption 3); since $\eta^h(R)$, $\epsilon^h(R)$, and $\frac{\tilde{c}^h(\tilde{R})}{\epsilon^h(\tilde{R})}$ are measurable (by the second-to-last step of the proof of Lemma 2); since this implies $\mathbb{E}_3 \left[ \tilde{c}^h(\tilde{R}) \right]$ is measurable; and since by Assumption 3, the elasticities are bounded and so integrable. To see that conditional expectations of $R(z_0^h)x_n^h$ conditional on $z_0^h$ (which is measurable by Assumption 3) exist, we argue that they are measurable and bounded by integrable functions; then dominated convergence implies they are integrable. Each $R(z_0^h)x_n^h$ is measurable since $w_h$ and $u_h$ are (locally) twice-continuously differentiable by the definition of a standard social objective and Assumption 2 since $c_0^h$ and $z_0^h$ are measurable by Assumption 4 and since elasticities are measurable by arguments in the second-to-last section of the proof of Lemma 2. Each $R(z_0^h)x_n^h$ is—by Assumption 4—bounded by a constant times one of the integrable functions $b_n(h)$ from the definition of local regularity.
For each variable $x_n^h$ referred to in the previous definition, we—for any $z \in \text{supp } g$—denote by $x(z)$ the ratio
\begin{equation}
  x(z) \equiv \frac{\mathbb{E}[R(z_0^h)x_n^h | z_0^h = z]}{R(z)}.
\end{equation}
for the remainder of the paper. Note that by Assumption 1, $x(z)$ is $n$-times continuously differentiable when the corresponding social objective is regular.

Under the additional structure imposed by a social objective’s regularity, not only individual contributions to welfare, but also aggregate welfare is defined and well-behaved locally to $R$.

**Lemma 5.** Suppose $(w^h)_{h \in H}, G)$ is a standard, locally regular social objective. Then there exists $\delta > 0$ such that $w^h \circ V^h(\hat{R})$ and its first two Frechet derivatives in $\hat{R}$ are bounded (as linear maps) across all $\hat{R} \in R + B_\delta(0)$ by linear combinations of the functions $b_n(h)$ from Definition 3 and, in particular, are $\mathcal{H}$-integrable.

**Proof.** See Appendix E.5.

**A.4 Other supporting concepts**

**A.4.1 Super-elasticity concepts**

Fix any $\delta > 0$ small enough that Lemma 2 applies. In the proof of Lemma 2, we establish that, at any $\hat{R} \in B_\delta(R)$, each household $h$’s compensated elasticity $\varepsilon^h(\hat{R})$ satisfies
\begin{equation}
  \varepsilon^h(\hat{R}) = \tilde{\varepsilon}^h(\tilde{z}(\hat{R}), \hat{R}), \quad \text{where} \quad \tilde{\varepsilon}^h(z, \hat{R}) \equiv \frac{1}{\frac{d \log M^h(\hat{R}(z), z)}{d \log z} - \frac{d \log \hat{R}^0(z)}{d \log z}} \quad (34)
\end{equation}
where $M^h(c, z) \equiv -\frac{u^h(c, z)}{u^h_z(c, z)}$ is $h$’s elasticity of substitution between consumption and leisure. One may use the expression $\tilde{\varepsilon}^h(z, \hat{R})$ in order to decompose changes in $h$’s elasticity with respect to taxes into two components: First, elasticity changes through the change in the tax schedule at a fixed labor supply. Second, $h$’s elasticity changes through $h$’s change in labor supply at a fixed tax schedule. One may divide the latter changes, in turn, into changes in elasticity stemming from income and compensated effects on labor supply. It is the latter that defines $\varepsilon^+(\hat{R})$. Formally,
\begin{equation}
  \varepsilon^+(\hat{R}) = \left[ \frac{d}{d \log z} \tilde{\varepsilon}^h(z, \hat{R}) \right] \tilde{\varepsilon}^h(\hat{R}). \quad (35)
\end{equation}

We show in the proof of Lemma 2 that the changes in elasticity contained in $\varepsilon^+(\hat{R})$ reflect two basic channels: The change in elasticity due to changes in the curvature of preferences as taxes change, and the change in elasticity due to changes in the curvature of the tax schedule as taxes change. That is,
\begin{equation}
  \varepsilon^+(z) \equiv \varepsilon_{+1}(z) + \frac{d^2 \log R(z)}{d \log z^2} \eta \varepsilon^2(z) + \frac{d^2 \log \hat{R}(z)}{d \log z^2} \varepsilon^3(z) \quad (36)
\end{equation}
The former are zero when preferences are (locally) additively CES, whereas the latter are zero when the tax schedule is (locally) CES.

\footnote{This is a valid definition since for any $z \in \text{supp } g \neq 0$, $R(z) > 0$ (see the proof of Lemma 2).}
B  Additional Discussion

B.1 Algebra for motivating example

Below, we briefly walk through the very straightforward algebra behind the example presented in Section 2.

Setup:

A unit measure $\mu$ of households $h \in H$ supply labor and consume in a static economy subject to a tax schedule $T$. The problem of each household $h$ is

$$V^h(T_{\bar{\tau}}) = \max_z z - T(z) - \frac{z^{1+\frac{1}{\beta h}}}{1+\frac{1}{\beta h}}(\theta^h)^{\frac{1}{\beta h}}$$  \hspace{1cm} (37)

We denote by $z^h(T_{\bar{\tau}})$ the maximizer of the household’s problem. Conditional on elasticity, productivity is distributed Pareto; $\theta^h | \beta^h \sim \text{Pareto}(\alpha > 1)$.

We assume the tax schedule $T$ is convex and initially imposes a constant top rate $\bar{\tau}_0$ on all incomes above some level $\bar{z}$.

Household labor supply

To begin, we characterize the labor supply problem (37) of each household $h$. We break this analysis into two cases. First, consider a household $h$ for whom $\bar{z} \geq \theta^h (1 - \bar{\tau}_0)$. Whenever the top tax rate is any $\bar{\tau} \geq \bar{\tau}_0$, $h$ must have an income $z$ weakly below $\bar{z}$ since otherwise decreasing $z$ increases utility at a rate

$$-1 + \bar{\tau} + \left(\frac{z}{\theta^h}\right)^{\frac{1}{\beta h}} \geq -1 + \bar{\tau}_0 + \left(\frac{z}{\theta^h}\right)^{\frac{1}{\beta h}} > 0.$$  \hspace{1cm} (38)

Moreover, note that the labor supply of any such household $h$ is unaffected by increases in the top tax rate, since $h$ already prefers some income $z < \bar{z}$.

Second, consider a household $h$ for whom $\bar{z} < \theta^h (1 - \bar{\tau}_0)$. At any top tax rate $\bar{\tau}$, $h$ must have an income $z$ weakly above $\bar{z}$ since otherwise—by $T$’s convexity—increasing $z$ increases utility at a rate weakly greater than

$$1 - \bar{\tau}_0 - \left(\frac{z}{\theta^h}\right)^{\frac{1}{\beta h}} > 0.$$  \hspace{1cm} (39)

An analogous argument implies that $h$ has income $z^h$ strictly above $\bar{z}$ at $\bar{\tau} > \bar{\tau}_0$ if and only if $\bar{z} < \theta^h (1 - \bar{\tau})$. When this latter inequality holds, the local differentiability of taxes and preferences implies the first order condition:

$$1 - \bar{\tau} = \left(\frac{z^h}{\theta^h}\right)^{\frac{1}{\beta h}}$$

In summary, household labor supply is given as a function of preferences and the top tax rate $\bar{\tau} \geq \bar{\tau}_0$ as

$$z^h(T_{\bar{\tau}}) = \begin{cases} \max \left[\theta^h (1 - \bar{\tau})^{\beta h}, \bar{z}\right] & \text{if } \theta^h (1 - \bar{\tau}_0)^{\beta h} > \bar{z} \\ z^h(T_{\bar{\tau}_0}) \leq \bar{z} & \text{otherwise.} \end{cases}$$  \hspace{1cm} (41)

Top incomes in each elasticity group:
We now compute, for any top tax rate \( \bar{\tau} \geq \bar{\tau}_0 \) the total income \( Z_{\text{top}}(\bar{\tau}|\beta) \) earned above \( \bar{z} \) by households of elasticity \( \beta \). We define this amount to be normalized by the size of the group, i.e. the number of households with elasticity \( \beta \).

\[
Z_{\text{top}}(\bar{\tau}|\beta) = \int \max \left[ \theta(1 - \bar{\tau})^\beta - \bar{z}, 0 \right] \text{density}(\theta|\beta) \, d\theta \\
= \int \max \left[ \theta(1 - \bar{\tau})^\beta - \bar{z}, 0 \right] \alpha^{-1} d\theta \\
= \alpha \int \frac{\theta(1 - \bar{\tau})^\beta - \bar{z}}{\bar{z}/(1 - \bar{\tau})^\beta} \theta^{-1} d\theta \\
= \alpha \left[ (1 - \bar{\tau})^\beta - \frac{1}{\alpha - 1} \int_{\bar{z}/(1 - \bar{\tau})^\beta}^\infty \theta^{-\alpha} d\theta - \bar{z} \int_{\bar{z}/(1 - \bar{\tau})^\beta}^\infty \theta^{-1} d\theta \right] \\
= \alpha \bar{z}^{1-\alpha} (1 - \bar{\tau})^{\alpha \bar{\beta}} \left[ \frac{1}{\alpha - 1} - \frac{1}{\alpha} \right] \\
= \frac{\bar{z}^{1-\alpha}}{\alpha - 1} (1 - \bar{\tau})^{\alpha \bar{\beta}} \\
\tag{42}
\]

**Tax revenue and its derivatives:**

Total tax revenue earned in the top bracket is simply the top tax rate times the income earned by each elasticity-group above \( \bar{z} \), or

\[
\text{Rev}_{\text{top}}(\bar{\tau}) = \mathbb{E}_\beta [ \bar{\tau} \cdot Z_{\text{top}}(\bar{\tau}|\beta) ] = k \cdot \mathbb{E}_\beta [ \bar{\tau} (1 - \bar{\tau})^{\alpha \bar{\beta}} ] , \tag{43}
\]

where \( \mathbb{E}_\beta[\cdot] \) is an expectation over \( \beta \) values according to their prevalence in the population.

Given the simple functional form for income earned in the top tax bracket, it is easy to compute derivatives of tax revenue as taxes increase:

\[
\text{Rev}'_{\text{top}}(\bar{\tau}) = \mathbb{E}_\beta [ k (1 - \bar{\tau})^{\alpha \bar{\beta}} ] - \bar{\tau} \mathbb{E}_\beta [ \alpha \beta k (1 - \bar{\tau})^{\alpha \bar{\beta} - 1} ] \\
= \mathbb{E}_\beta [ Z_{\text{top}}(\bar{\tau}|\beta) ] - \frac{\bar{\tau}}{1 - \bar{\tau}} \alpha \mathbb{E}_\beta [ \beta Z_{\text{top}}(\bar{\tau}|\beta) ] \\
\text{Rev}''_{\text{top}}(\bar{\tau}) = -\mathbb{E}_\beta [ \alpha \beta k (1 - \bar{\tau})^{\alpha \bar{\beta} - 1} ] - \mathbb{E}_\beta [ \alpha \beta k (1 - \bar{\tau})^{\alpha \bar{\beta} - 1} ] \\
+ \bar{\tau} \mathbb{E}_\beta [ \alpha \beta (\alpha \beta - 1) k (1 - \bar{\tau})^{\alpha \bar{\beta} - 2} ] \\
= -\frac{2\alpha}{1 - \bar{\tau}} \mathbb{E}_\beta [ \beta Z_{\text{top}}(\bar{\tau}|\beta) ] - \frac{\bar{\tau} \alpha}{(1 - \bar{\tau})^2} \mathbb{E}_\beta [ \beta Z_{\text{top}}(\bar{\tau}|\beta) ] \\
+ \frac{\bar{\tau} \alpha^2}{(1 - \bar{\tau})^2} \mathbb{E}_\beta [ \beta^2 Z_{\text{top}}(\bar{\tau}|\beta) ] \\
= \frac{\alpha \mathbb{E}_\beta [ Z_{\text{top}}(\bar{\tau}|\beta) ]}{(1 - \bar{\tau})^2} \left[ -(2 - \bar{\tau}) \mathbb{E}_{\text{top}} [ \beta ] + \tau \alpha \mathbb{E}_{\text{top}} [ \beta^2 ] \right] \\
\tag{44}
\]

where \( \mathbb{E}_{\text{top}}[\cdot] \) is an expectation over elasticity groups that weights each proportionally to the share of income earned in the top bracket by households with that elasticity.
Welfare and its derivatives:

Finally, we compute the welfare of each top-earner and its derivatives with respect to \( \bar{\tau} \).

Plugging in our expression for incomes \( z^h(T_{\bar{\tau}}) \) into the household utility function, we obtain that for all \( h \) with \( z^h(T_{\bar{\tau}_0}) > \bar{z} \) and \( \bar{\tau} \) near enough to \( \bar{\tau}_0 \),

\[
V^h(T_{\bar{\tau}}) \equiv \max_{z} \left( \frac{z - T(z)}{1 + \beta h} - \frac{z^{1 + \frac{1}{\beta h}}}{1 + \beta h} \right) = \bar{\tau} \bar{z} - T(\bar{z}) - \frac{\theta^h(1 - \bar{\tau})^{1 + \beta h}}{1 + \beta h}.
\]

(45)

We may therefore differentiate:

\[
\frac{d}{d\bar{\tau}} \bigg|_{\bar{\tau} = \bar{\tau}_0} V^h(T_{\bar{\tau}}) = \bar{z} - \theta^h(1 - \bar{\tau}_0)^{\beta h} = -(z^h(T_{\bar{\tau}_0}) - \bar{z})
\]

\[
\frac{d^2}{d\bar{\tau}^2} \bigg|_{\bar{\tau} = \bar{\tau}_0} V^h(T_{\bar{\tau}}) = \beta^h \theta^h(1 - \bar{\tau}_0)^{\beta h - 1} > 0
\]

(46)

Of course, the utilities of households with initial incomes below \( \bar{z} \) are not affected by increases in \( \bar{\tau} \).

We conclude that

\[
\frac{d}{d\bar{\tau}} \bigg|_{\bar{\tau} = \bar{\tau}_0} \tilde{\lambda} \cdot W_{\text{top}}(\bar{\tau}) = -\tilde{\lambda} \cdot \mathbb{E}_\beta[Z_{\text{top}}(\bar{\tau} | \beta)] \quad \text{and} \quad \frac{d^2}{d\bar{\tau}^2} \bigg|_{\bar{\tau} = \bar{\tau}_0} \tilde{\lambda} \cdot W_{\text{top}}(\bar{\tau}) \geq 0,
\]

(47)

as we have used in the main text.

B.2 One-dimensional heterogeneity: “shift and exploit”

[2007] shows that if households differ only along a single dimension \( \theta^h \) and, for some common concave function \( v_c \) and convex function \( v_z \), both increasing, have utility

\[
U^h(c, z) = v_c(c) - v_z(z) / \theta^h,
\]

(48)

then the problem of Pareto planner is globally convex. A similar result holds in our setting: The problem of any planner with a standard objective so long as the weighting functions \( u^h \) are concave. In particular, the first-order condition \( \text{[ABC]} \) implies the second-order condition \( \text{[DEFG]} \).

A natural question is whether this convexity is an essential feature of one-dimensional settings or a consequence of the function form \( \text{[48]} \). To answer this question, we consider the planner’s second-order condition in the general one-dimensional case where there is a unique type \( h(z) \) who earns each income \( z \in \text{supp} g \). Proposition \( \text{[C.3]} \) in Appendix \( \text{[C.3]} \) leverages the following two special properties of \( \text{[48]} \) in order to show that—for these preferences—\( \text{[ABC]} \) implies \( \text{[DEFG]} \):

- The marginal rate of substitution between consumption and leisure is weakly increasing in consumption and decreasing in leisure\(^{70}\)

\(^{69}\) See Proposition \( \text{[C.3]} \).

\(^{70}\) This is different than what is commonly referred to as "decreasing marginal rates of substitution",...
• All of the variation in compensated elasticity across income levels is due to differences in the income and consumption levels at which households’ preferences are evaluated, rather than differences in the curvature of their preferences at a given level of income and consumption.

The first property is quite weak and satisfied by the standard functional forms in the literature. By contrast, the second property is a much more “special” feature. The following two examples illustrate how the planner may improve taxes when this knife-edge assumption fails by a significant enough amount.

**Example 1.** Taxes are linear, i.e. \( R(z) = rz \) for \( r \in (0,1) \). A one-dimensional continuum of households \( h \) have additive-CES preferences with idiosyncratic elasticities \( \beta(\theta^h) \):

\[
\begin{align*}
    u^h(c, z) &= c + \frac{z^{1+1/\beta(\theta^h)}}{1 + 1/\beta(\theta^h)} / \theta^h
\end{align*}
\]

Finally, suppose \( \beta(\theta) \) is very sharply decreasing around some type \( \theta^h(z^*) \).

**Example 2.** Taxes are linear, i.e. \( R(z) = rz \) for \( r \in (0,1) \). A one-dimensional continuum of households \( h \) have preferences:

\[
\begin{align*}
    u^h(c, z) &= \log(c) - v\left(\frac{z}{\theta^h}\right)
\end{align*}
\]

for some increasing and concave function \( v \) with variable elasticity. It is easy to verify that this implies each household supplies labor \( z^h = \theta^h n^* \) for some common \( n^* \) and has a (common) compensated elasticity

\[
\varepsilon^h(R) = \frac{1}{1 + \frac{d\log v'(n^*)}{d\log n^*}}.
\]

Finally, suppose \( \frac{d\log v'(n^*)}{d\log n^*} \) is very sharply decreasing locally to \( n^* \).

We now explain why—in either example—any planner can increase the value of her Lagrangian using a narrow variation in taxes around \( z^* \), as in Figure 2. To first order, this variation changes the elasticity at each income level. In Example 1, this change occurs because each household adjusts its income, shifting which elasticities are present at each income level. This effect is strongest where the change in taxes is the most steeply increasing (decreasing), where households with much higher (lower) elasticities are drawn in from the left (right). In Example 2, this change occurs because each household adjusts its elasticity. Namely, those whose labor supplies increase (decrease) also experience increased (decreased) elasticities, due to the changing curvature of \( v'(\cdot) \). In either example, the effect of the shift in elasticities is that the same tax variation has much better behavioral effects if performed a second time: Elasticities are particularly high where marginal taxes increase the most and particularly low where marginal taxes decrease the most. More succinctly, the planner can shift elasticities with a first tax change and then exploit this shift with a second. If the shift in elasticities is dramatic enough, this effect causes the two deviations to improve the planner’s Lagrangian on net.

which applies to shifts along an indifference curve. The condition we study here is slightly stronger than the convexity of preferences.

\[^{71}\text{In the cases where it does fail—such as when households have certain non-convex preferences—there seems to be little reason to hope the planner’s problem should be convex, anyways.}\]
B.3 Simulation exercise

We now describe in detail the simulation exercise alluded to in Section 6. Concretely, each household $h$ has constant compensated elasticities $\beta^h$ and no income elasticity. Within every income level 80% of households have $\beta^h = 0.01$ and 20% have $\beta^h = 2$. We roughly approximate the distribution of income in our data by assuming that productivity is distributed according to a generalized Pareto distribution with location $0$, scale $e^9$, and shape $1/2$ (this implies a Pareto tail of shape 2). Marginal taxes are initially 50% at every income level. The planner places a constant but not common welfare weight on each household, is indifferent between households with the same initial income, and ensures that her first-order condition holds at the initial tax schedule. Under this calibration, the planner’s first-order condition holds exactly but the second-order condition fails for incomes above $\approx 90,000$.

Next, we perturb marginal taxes slightly, lowering them by 1% at every income level. After computing how households respond to this change, we recompute the planner’s first-order condition, which no longer holds. We then move taxes slightly in the direction in which the first-order condition fails. This does not simply push taxes back to where they began, as—at high incomes—taxes were initially at a local minimum. We iterate this procedure until it converges.

Figure 11 shows how marginal retention evolves between the initial tax schedule and the final schedule to which our procedure converges. Notably, taxes converge to a new, much lower marginal rate at high incomes—where the second-order condition initially failed—but are barely changed at lower incomes—where the second order condition initially held. Intuitively, an increase in marginal retention sorts high-elasticity households into higher income levels, rationalizing further decreases in marginal taxes at the top and so on until eventually taxes are high enough.

C Proofs of main results

This section contains proofs of our main results. These proofs focus on the main conceptual steps and relegate many supporting details to Appendix E.

C.1 Proof of Theorem 1

The proof has three main steps. First, Lemma 6 computes the first- and second-order derivatives of aggregate tax revenue. Second, Lemma 7 does the same for aggregate welfare. Third, we use these derivatives to study the planner’s first- and second-order necessary conditions for optimality of the tax schedule.

**Lemma 6.** Take $\Delta \in \Delta$ and suppose $\Delta$ is non-zero only on some interval $[\bar{z}, \overline{z}] \subset \text{supp } g$. Then

\[
\begin{align*}
D_{\Delta} \int \left( z^h(R) - R(z^h(R)) \right) d\mu &= \int_{\text{supp } g} g(z) \psi(z) \Delta(z) dz \\
D_{\Delta \Delta} \int \left( z^h(R) - R(z^h(R)) \right) d\mu &= \int_{\text{supp } g} g(z) \left[ \Psi_0(z) \left( \frac{\Delta(z)}{R(z)} \right)^2 + \Psi_1(z) \left( \frac{\Delta^2(z)}{R(z)} \right) \right] dz \\
\text{where } \psi(z) &= \frac{1 - R'(z)}{R(z)} \left( \frac{d \log R(z)}{d \log z} g(z) + \alpha(z) - \frac{d \log g(z)}{d \log z} \right) - 1 \\
\Psi_1(z) &= -z \left( 1 + R'(z) \right) \varepsilon(z) + z \left( 1 - R'(z) \right) \left( \alpha(z) - \frac{d \log g(z)}{d \log z} \right) \varepsilon(z) + \varepsilon^+(z)
\end{align*}
\]
Figure 11: Schedule of marginal retention rates following a small perturbation away from initial taxes in the Laffer valley.

and $\psi(z)$, $\Psi_0(z)$, and $\Psi_1(z)$ are continuous functions of $z$ on $\text{supp } g$. ($\Psi_0(z)$ is defined in the proof.)

Proof. Let $f^h(\tilde{R}) \equiv z^h(\tilde{R}) - \tilde{R}(z^h(\tilde{R}))$ denote the tax revenue earned from each household $h \in H$. In Appendix E.6.1 we establish that for some $\delta > 0$, aggregate tax revenue $\int f^h(\tilde{R})d\mu$ is defined and has two continuous Frechet derivatives at all $\tilde{R} \in R + B_\delta(0)$. Moreover, these derivatives satisfy

$$D^n \int f^h(\tilde{R})d\mu = \int D^n f^h(\tilde{R})d\mu. \quad (53)$$

for $n = 1, 2$.

We now proceed to compute these derivatives.

First derivative of tax revenue

Fix any $\Delta \in \Delta$ satisfying $\Delta(z) = 0$ for all $z$ outside of some interval $[\bar{z}, \overline{z}] \subset \text{supp } g$. To compute the first derivative of tax revenue, we combine (53) with the expression (122) for $D_\Delta z^h(R)$ in the proof of Lemma 272

$$D_\Delta \int f^h(\tilde{R})d\mu = \int \left[ (1 - R'(z^h_0)) \frac{\Delta(z^h_0)}{R(z^h_0)} + \epsilon'(R) \frac{\Delta'(z^h_0)}{R'(z^h_0)} - \Delta(z^h_0) \right] d\mu \quad (54)$$

72 Appendix E.6.2 walks through the measure-theoretic steps used below to move between the first and second line; we use similar steps without explicit reference for the rest of the proofs.
Finally, we integrate by parts in order to convert the term proportional to $\Delta'(z)$ into a term proportional to $\Delta(z)$. Here, we use that (a) $\Delta(z) = 0$ outside of $[z, \bar{z}] \subset \text{supp } g$, (b) by $R'(z)$’s continuity (from Assumption 1) and the argument in the proof of Lemma 2 that $R'(z) > 0$ at all $z > 0$, $R'(z)$ is bounded above zero on $[z, \bar{z}]$, (c) by Assumptions 1 and 3 and the definition of $\Delta$, $g(z) \frac{1-R'(z)}{R'(z)} \varepsilon(z)z$ and $\Delta(z)$ are continuously differentiable.

\[
\int_\zeta^\bar{z} g(z) \frac{1-R'(z)}{R'(z)} \varepsilon(z) z\Delta'(z) dz
= - \int_\zeta^\bar{z} g(z) \frac{1-R'(z)}{R'(z)} \varepsilon(z) \left( -\alpha(z) + \frac{d\log R(z)}{d\log z} \left( \frac{1-R'(z)}{R'(z)} \right) + \frac{d\log \varepsilon(z)}{d\log z} \right) \Delta(z) dz.
\]

(55)

Substituting this in gives us the expression $\psi(z)$ in the statement of the lemma.

Finally, the continuity of $\psi(z)$ on $\text{supp } g$ follows from Assumptions 1 and 4 and the fact, noted above, that $R'(z) > 0$ for all $z \in \text{supp } g \not\equiv 0$.

**Second derivative of tax revenue**

To begin, note that simplifying the expression (130) for $D^2_{\Delta} \tilde{R}(z^h(\tilde{R}))$ in the proof of Lemma 3 and combining it with $D^2_{\Delta} z^h(R) = z^h [(D_{\Delta} \log z^h(R))^2 + D^2_{\Delta} \log z^h(R)]$ gives us

\[
D^2_{\Delta} f^h(R) = z^h \left[ \frac{1-R''(z)}{R''(z)} \right] \left( \frac{D_{\Delta} \log z^h(R)}{R(z)} + \frac{D^2_{\Delta} \log z^h(R)}{R''(z)} \right) R'(z)
- \frac{d\log R'(z)}{d\log z} \left( \frac{D_{\Delta} \log z^h(R)}{R''(z)} \right) D(z) \Delta'(z) + \frac{d\log R''(z)}{d\log z} \left( \frac{D_{\Delta} \log z^h(R)}{R''(z)} \right) D'(z) \Delta''(z)
\]

(56)

Substituting in for $D_{\Delta} \log z^h(R)$ and $D^2_{\Delta} \log z^h(R)$ using the expressions (122) and (130) in the proof of Lemma 2, employing (53), and finally changing variables to integrate over income rather than households, we obtain

\[
D^2_{\Delta} \int f^h(R) d\mu = \int_{\text{supp } g} g(z) \left[ A(z) \left( \frac{\Delta(z)}{R(z)} \right)^2 + B(z) \left( \frac{\Delta(z) \Delta'(z)}{R(z) \Delta(z)} \right) + C(z) \left( \frac{\Delta'(z)}{R(z)} \right)^2 \right.
+ \left. D(z) \left( \frac{\Delta'(z)}{R(z) \Delta'(z)} \right) + E(z) \left( \frac{\Delta'(z) \Delta''(z)}{R'(z) \Delta'(z)} \right) \right] dz,
\]

where

\[
A(z) = \left( \frac{\Delta(z)}{R(z)} \right)^2 + \left( \frac{\Delta(z) \Delta'(z)}{R(z) \Delta(z)} \right) + C(z) \left( \frac{\Delta'(z)}{R(z)} \right)^2
\]

\[
B(z) = 2z \left( 1-R'(z) \right) \left[ \frac{d\log R(z)}{d\log z} \eta^2(z) + \frac{d\log R'(z)}{d\log z} \eta^2(z) \right] + \frac{d\log R''(z)}{d\log z} \left( \eta^2(z) + \eta z \right) - z R'(z) \left( \frac{d\log R'(z)}{d\log z} \eta^2(z) \right)
\]

\[
C(z) = \left( \frac{\Delta'(z)}{R(z) \Delta'(z)} \right) + 2z \left( 1-R'(z) \right) \left[ \eta z + \frac{d\log R'(z)}{d\log z} \right] \left( \eta z \right) + \frac{d\log R''(z)}{d\log z} \left( \eta^2(z) + \eta + 1 \right) - 2z R'(z) \left( \frac{d\log R'(z)}{d\log z} \eta^2(z) \right)
\]

\[
D(z) = \frac{d\log R'(z)}{d\log z} \left( \eta^2(z) + \eta z \right) + \frac{d\log R''(z)}{d\log z} \left( \eta^2(z) + \eta + 1 \right) - z R'(z) \left( \frac{d\log R'(z)}{d\log z} \eta^2(z) \right)
\]

\[
E(z) = 2z \left( 1-R'(z) \right) \left( \eta z \right)
\]

(57)

Our assumptions guarantee that $A(z)$ and $C(z)$ are continuous, $B(z)$ and $E(z)$ are con-
continuously differentiable, and \( D(z) \) is twice-continuously differentiable in \( z \) on \( \text{supp} \, g \),
and moreover that each additive term of \((57)\) is integrable in isolation.\(^{75}\)

In order to reach the expression in the statement of the Lemma, we integrate by parts.\(^{76}\)

\[
\int_{\text{supp} \, g} g(z) B(z) \frac{\Delta(z) \Delta'(z)}{R(z) R'(z)} \, dz = - \int_{\text{supp} \, g} \frac{d}{dz} \left( \frac{g(z) B(z)}{R(z) R'(z)} \right) \frac{1}{2} \Delta(z)^2 \, dz
\]

\[
\int_{\text{supp} \, g} g(z) E(z) \frac{\Delta'(z) \Delta''(z)}{R(z) R'(z)} \, dz = - \int_{\text{supp} \, g} \frac{d}{dz} \left( \frac{g(z) E(z)}{R(z) R'(z)} \right) \frac{1}{2} \Delta(z)^2 \, dz
\]

\[
\int_{\text{supp} \, g} g(z) D(z) \frac{\Delta(z) \Delta''(z)}{R(z) R'(z)} \, dz = - \int_{\text{supp} \, g} \frac{d}{dz} \left( \frac{g(z) D(z)}{R(z) R'(z)} \right) \Delta(z) \Delta'(z) \, dz - \int_{\text{supp} \, g} \frac{g(z) z D(z)}{R(z) R'(z)} \Delta'(z)^2 \, dz
\]

We conclude that

\[
D^2_{\Delta \Delta} \int (z^h(R) - R(z^h(R))) \, d\mu = \int_{\text{supp} \, g} g(z) \left[ \Psi_0(z) \left( \frac{\Delta(z)}{R(z)} \right)^2 + \Psi_1(z) \left( \frac{\Delta'(z)}{R'(z)} \right)^2 \right] \, dz
\]

where \( \Psi_0(z) = \frac{1}{2} [A(z) - \frac{R(z)^2}{2g(z)} \frac{d}{dz} \left( \frac{g(z) B(z)}{R(z) R'(z)} \right)] \) and \( \Psi_1(z) = \frac{1}{2} [C(z) - \frac{2 \log R(z)}{2g(z)} \frac{d}{dz} \left( \frac{g(z) z E(z)}{R'(z)^2} \right)] \).

The continuity of \( \Psi_0 \) and \( \Psi_1 \) on \( \text{supp} \, g \) follows from our earlier observations about \( A(z), ..., E(z) \), Assumptions 1 and 5, and the fact that \( R(z), R'(z) > 0 \) on \( \text{supp} \, g \) (see the proof of Lemma 2).

To complete the proof, it remains to simplify the expression for \( \Psi_1(z) \). Here, the main step is to compute the \( E(z) \) term. Since by definition \( g(z) \not\equiv 0 \) for \( z \in \text{supp} \, g \), we have

\[
- \frac{R(z)^2}{2} \frac{d}{dz} \left( \frac{g(z) z E(z)}{R'(z)^2} \right) = -g(z) \frac{1}{2} \left[ -\alpha(z) E(z) + z E'(z) - 2 \frac{d \log R'(z)}{d \log z} E(z) \right]
\]

\[
= g(z) \left[ z(1 - R'(z)) \varepsilon^2(z) \left( \alpha(z) + 2 \frac{d \log R'(z)}{d \log z} - 1 - \frac{d \log R(z)}{d \log z} \right) + z R'(z) \frac{d \log R'(z)}{d \log z} \varepsilon^2(z) \right]
\]

Substituting this expression, the definitions of \( C(z) \) and \( D(z) \), and the definition \( \varepsilon^+(z) \equiv \frac{d}{d \log z} \left( \eta \varepsilon^2(z) \right) + \frac{d}{d \log z} \left( \frac{d \log R'(z)}{d \log z} \right) \varepsilon^3(z) + \varepsilon^+(z) \) (see Appendix A.4.1) into the definition of \( \Psi_1(z) \) and cancelling terms gives the expression in the statement of the lemma.

\[\blacksquare\]

**Lemma 7.** Let \((w^h)_{h \in H}, G)\) be a standard, regular social objective. Take \( \Delta \in \Delta \) and

73 This follows from Assumptions 1 and 5 and the facts that (a) as shown in the proof of Lemma 2 \( R(z), R'(z) > 0 \) at all \( z > 0 \), and (b) by Assumption 3 \( 0 \not\in \text{supp} \, g \).

74 Integrability is immediate from the continuity discussed above, the continuity and positivity of \( R(z) \) and \( R'(z) \), the continuity of \( \Delta(z) \) implied by the definition of \( \Delta \), and the fact that \( \Delta(z) \) is zero outside of \([z, \bar{z}]\).

75 The validity of each integration by parts follows from that (a) since \( \Delta \) is zero outside of \([z, \bar{z}]\), we may restrict each integral to that interval (b) \( \Delta(z) \) is zero at the endpoints of the interval, and (c) since \([z, \bar{z}] \subset \text{supp} \, g \), similar continuity arguments to those above ensure each term of each integrand is continuously differentiable as needed.
suppose \( \Delta \) is non-zero only on some interval \( [z, \bar{z}] \subset \text{supp} \, g \). Then

\[
D_\Delta \int w^h \circ V^h(R) d\mu = \int_{\text{supp} \, g} g(z) \lambda(z) \Delta(z) dz,
\]

\[
D^2_{\Delta \Delta} \int w^h \circ V^h(R) d\mu = \int_{\text{supp} \, g} g(z) \left[ \Phi_0(z) \left( \frac{\Delta(z)}{R(z)} \right)^2 + \Phi_1(z) \left( \frac{\Delta'(z)}{R'(z)} \right)^2 \right] dz
\]

where \( \Phi_0(z) \equiv R(z) \left( \lambda_\gamma(z) + \frac{d \log R(z)}{d \log z} \left( \frac{d^2 \lambda}{d \log z} \right) (z) + \frac{1}{2} \left( \alpha(z) + \frac{d \log R(z)}{d \log z} \right) (\lambda \eta(z) - \frac{1}{2} z (\lambda \eta'(z)) \right) \)

\( \Phi_1(z) \equiv R(z) \frac{d \log R(z)}{d \log z} (\lambda \epsilon(z)) \)

where \( \Phi_0(z) \) and \( \Phi_1(z) \) are continuous functions of \( z \) on \( \text{supp} \, g \).

**Proof.** Let \( f^h(\bar{R}) \equiv w^h \circ V(\bar{R}) \). In Appendix E.6.1 we establish that for some \( \delta > 0 \), aggregate welfare \( \int f^h(\bar{R}) d\mu \) is defined and has two continuous Frechet derivatives at all \( \bar{R} \in \bar{R} + B_\delta(0) \). Moreover, these derivatives satisfy

\[
D^n \int f^h(\bar{R}) d\mu = \int D^n f^h(\bar{R}) d\mu.
\]

for \( n = 1, 2 \).

**First derivative of welfare**

Fix any \( \Delta \in \Gamma \) satisfying \( \Delta(z) = 0 \) for all \( z \) outside of some interval \( [z, \bar{z}] \subset \text{supp} \, g \).

To compute the first derivative of tax revenue, we combine (62) with the expression (137) for \( D_\Delta w^h \circ V^h(\bar{R}) \) in the proof of Lemma 5.

\[
D_\Delta \int f^h(R) d\mu = \int_{\text{supp} \, g} R(z_0^h) (w^h \circ u^h)_c \left( c^h_0, z_0^h \right) \frac{\Delta(z_0^h)}{R(z_0^h)} d\mu = \int_{\text{supp} \, g} g(z) \lambda(z) \Delta(z) dz
\]

where \( \lambda(z) \) is the average marginal value of transfers to \( z \)-earners relative to tax revenue, as defined in and below Definition 5.

**Second derivative of welfare**

To begin, we simplify the expression (138) for \( D^2_{\Delta \Delta} w^h \circ V^h(R) \) in the proof of Lemma 5.

\[
D^2_{\Delta \Delta} f^h(R) = (w^h \circ u^h)_c \left( c^h_0, z_0^h \right) (c^h_0)^2 \left( \frac{\Delta(z_0^h)}{R(z_0^h)} \right)^2 + (w^h \circ u^h)_c \left( c^h_0, z_0^h \right) c^h_0 \frac{d \log R(z_0^h)}{d \log z} 1 \frac{\Delta \log z^h(R)}{\epsilon^h(R)} \left( \frac{1}{\epsilon^h(R)} \right)^2.
\]

Substituting in for \( D_\Delta \log z^h(R) \) using the expression (122) in the proof of Lemma 2, employing (62), and finally changing variables to integrate over income rather than house-
holds, we obtain
\[
D^2_{\Delta \Delta} \int f(R) d\mu = \int_{\text{supp } g} g(z) \left[ A(z) \left( \frac{\Delta(z)}{R(z)} \right)^2 + B(z) \frac{\Delta(z) \Delta'(z)}{R(z) R'(z)} + C(z) \left( \frac{\Delta'(z)}{R'(z)} \right)^2 \right] dz.
\]
where
\[
A(z) \equiv R(z) \left( (\lambda_\gamma)(z) + \frac{d \log R(z)}{d \log z} \left( \frac{\eta^2}{z} \right) (z) \right)
\]
\[
B(z) \equiv R(z)(\lambda \eta)(z) \frac{d \log R(z)}{d \log z}
\]
\[
C(z) \equiv R(z) \frac{d \log R(z)}{d \log z} (\lambda \epsilon)(z)
\]
where the various terms \((\lambda x)(z)\) are as defined in and below Definition 3. Our assumptions guarantee that \(A(z)\) and \(C(z)\) are continuous and \(B(z)\) is continuously differentiable in \(z\) on \(\text{supp } g\) and moreover that each additive term of (65) is integrable in isolation.

In order to reach the expression in the statement of the Lemma, we integrate by parts:
\[
\int_{\text{supp } g} g(z) B(z) \frac{\Delta(z) \Delta'(z)}{R(z) R'(z)} dz = - \int_{\text{supp } g} \frac{d}{dz} \left( \frac{g(z) B(z)}{R(z) R'(z)} \right) \frac{1}{2} \Delta(z)^2 dz
\]  
(66)

We conclude that
\[
D^2_{\Delta \Delta} \int (z^h(R) - R(z^h(R))) d\mu = \int_{\text{supp } g} g(z) \left[ \Phi_0(z) \left( \frac{\Delta(z)}{R(z)} \right)^2 + \Phi_1(z) \left( \frac{\Delta'(z)}{R'(z)} \right)^2 \right] dz
\]  
(67)

where
\[
\Phi_0(z) \equiv A(z) - \frac{R(z)^2}{2g(z)} \frac{d}{dz} \left( \frac{g(z) B(z)}{R(z) R'(z)} \right) \quad \text{and} \quad \Phi_1(z) \equiv C(z)
\]

The continuity of \(\Phi_0\) and \(\Phi_1\) on \(\text{supp } g\) follows from our earlier observations about \(A(z), B(z), C(z)\), Assumptions 1 and 5, Definition 3, and the fact that \(R(z), R'(z) > 0\) on \(\text{supp } g\) (see the proof of Lemma 2). To complete the proof, it remains to simplify the expression for \(\Phi_0(z)\) by expanding the \(B(z)\) term. Since, by definition \(g(z) \neq 0\) for \(z \in \text{supp } g\), we have
\[
- \frac{R(z)^2}{2g(z)} \frac{d}{dz} \left( \frac{g(z) B(z)}{R(z) R'(z)} \right) = - \frac{1}{g(z)} \frac{R(z)^2}{2z} \frac{d}{d\log z} \left[ g(z)z(\lambda \eta)(z) \right]
\]
\[
= \frac{1}{g(z)} \frac{R(z)^2}{2z} \left[ g(z)z(\lambda \eta)(z) \right] (R(z) R'(z))^{-1} - g(z) \frac{z(\lambda \eta)(z)}{R(z)} \frac{d \log R(z)}{d \log z}
\]
\[
= \frac{R(z)}{2} \left[ \left( z(\lambda \eta)(z) - z(\lambda \eta)'(z) \right) \right]
\]  
(68)

Simplifying this expression and substituting it into the definition of \(\Phi_0(z)\) gives the expression in the statement of the Lemma.

\[\square\]

We now turn to the third step of the proof of Theorem 1. Here, the main idea is to

---

76 This follows from Assumptions 1 and 5, Definition 3, and the facts that (a) as shown in the proof of Lemma 2, \(R(z), R'(z) > 0\) at all \(z > 0\), and (b) by Assumption 3, \(0 \notin \text{supp } g\).

77 Integrability is immediate from the continuity discussed above, the continuity and positivity of \(R(z)\) and \(R'(z)\), the continuity of \(\Delta(z)\) implied by the definition of \(\Delta\), and the fact that \(\Delta(z)\) is zero outside of \([\underline{z}, \overline{z}]\).

78 The validity of integration by parts follows from that (a) since \(\Delta\) is zero outside of \([\underline{z}, \overline{z}]\), we may restrict each integral to that interval (b) \(\Delta(z)\) is zero at the endpoints of the interval, and (c) since \([\underline{z}, \overline{z}] \subset \text{supp } g\), similar continuity arguments to those above ensure each term of each integrand is continuously differentiable as needed.
use the derivatives computed in Lemmas 6 and 7 to study the first- and second-order necessary conditions that must hold for the planner who rationalizes \( R \).

More concretely, we may—since by assumption \( R \) is locally rationalized by a standard, regular social objective—take \((w^h)_{h \in H}, G\) to be such a social objective. Since \( R \) is locally rationalized by the social objective, \( 0 \) must solve:

\[
0 \in \arg \max_{\Delta \in \Delta} F(\Delta) \quad \text{s.t.} \quad H(\Delta) \in \mathbb{R}_{\geq 0} \\
\text{where } F(\Delta) \equiv \begin{cases} 
\int w^h \circ u^h \left( (R + \Delta)(z^h(R + \Delta)) \right) d\mu, & \text{if } \Delta \in B_\delta(0) \\
F(0) - 1 & \text{if } \Delta \not\in B_\delta(0)
\end{cases} \\
H(\Delta) \equiv \begin{cases} 
\int \left[ z^h(R + \Delta) - (R + \Delta)(z^h(R + \Delta)) \right] d\mu - G, & \text{if } \Delta \in B_\delta(0) \\
0, & \text{if } \Delta \not\in B_\delta(0)
\end{cases}
\tag{69}
\]

where \( \delta > 0 \) is small enough that \( F \) and \( H \) are well-defined and within \( B_\delta(0) \) have well-defined and continuous first and second Frechet derivatives (see Appendix E.6.1).

In Appendix E.6.3 we show that the optimization problem (69) satisfies the conditions required to apply standard results from optimization theory on Banach spaces. In particular, the fact that \( 0 \) solves (69) implies: If \( DH(0) \neq 0 \), then there exists \( \kappa \in \mathbb{R}_{\geq 0} \) such that

- A first-order condition holds: \( DF(0) + \kappa DH(0) = 0 \)
- A second-order condition holds: for all non-zero \( \Delta \in \Delta \) satisfying \( D_\Delta H(0) = 0 \), \( D^2_\Delta F(0) + \kappa D^2_\Delta H(0) \leq 0 \).

Since the case where \( DH(0) = 0 \)—i.e. tax revenue is, to first-order, invariant to tax changes—is unlikely to apply in practice, we relegate it to Appendix E.6.4. We show that in this case \( \text{ABC} \) holds with equality for all \( z \in \text{supp } g \), so the theorem holds. The remainder of the proof focuses on the complementary case where \( DH(0) \neq 0 \).

First-order condition

Recall that there exists \( \kappa \geq 0 \) for which \( DF(0) + \kappa DH(0) = 0 \). As we have assumed \( DH(0) \neq 0 \), this implies there exists \( \Delta \in \Delta \) with either \( D_\Delta F(0) \neq 0 \), so we must have \( \kappa > 0 \). Since, by the definition of a standard social objective, \( \lambda(z) \geq 0 \) for all \( z \), Lemma 6 implies \( D_\Delta F(0) \geq 0 \) for all \( \Delta \in \Delta \) satisfying \( \Delta(z) \geq 0 \) for all \( z \in \text{supp } g \).

Putting together these observations with the expression for the first derivative of revenue in Lemma 6, we have that for all \( \Delta \in \Delta \) satisfying \( \Delta(z) \geq 0 \) for all \( z \in \text{supp } g \),

\[
D_\Delta H(0) = \int_{z \in \text{supp } g} g(z)\psi(z)\Delta(z)dz \geq 0, \tag{70}
\]

where \( \psi(z) \) is as in Lemma 6 and recall \( \psi(z) \) is continuous on \( \text{supp } g \).

We conclude that \( \psi(z) \geq 0 \) for all \( z \in \text{supp } g \), i.e. the first part of the theorem holds. Otherwise, the continuity of \( g(z) \) (from Assumption 5) and \( \psi(z) \) imply there exists an interval \([\underline{z}, \overline{z}] > \underline{z}] \subset \text{supp } g\) so that i.e. \( g(z)\psi(z) < 0 \) at all \( z \in [\underline{z}, \overline{z}] \). The result then follows from considering any weakly positive function \( \Delta(z) \) that is strictly positive on a non-zero-measure sub-interval of \([\underline{z}, \overline{z}]\), zero outside of \([\underline{z}, \overline{z}]\), and is contained in \( \Delta \)\(^{79}\)

\(^{79}\text{We give an example in Appendix E.6.5}\)
Second-order condition

We now argue that the second-order condition stated above implies \( \Psi_1(z) \leq 0 \) for all \( z \in \text{supp} g \)—where \( \Psi_1 \) is as defined in Lemma 6—as claimed in the statement of the theorem. It suffices to show \( \Phi_1(z) + \kappa \Psi_1(z) \leq 0 \) for all \( z \in \text{supp} g \)—where \( \Psi_0 \) is as defined in Lemma 7—since \( \kappa > 0 \) and since \( \Phi_1(z) \geq 0 \) because (a) \( R(z), R'(z) > 0 \) (see the proof of 2) and (b) by Assumptions 2 the definition of a standard social objective, and the observation in the proof of 2 that \( \varepsilon^h(R) > 0 \), we have \( \varepsilon^h(R), \lambda^h(R) \geq 0 \) for all \( h \in \mathcal{H} \).

To see this, suppose not, i.e. \( \Phi_1(z^*) + \kappa \Psi_1(z^*) > 0 \) at some \( z^* \in \text{supp} g \). By the continuity established in Lemma 6 and 7 as well as that of \( g(z) \) from Assumption 3 there then exists an interval \([\hat{z}^*, \bar{z}] > \hat{z} \subset \text{supp} g\) on which \( g(z)(\Phi_1(z) + \kappa \Psi_0(z)) \) is bounded above zero and \( g(z)(\Phi_0(z) + \kappa \Psi_0(z)) \) and \( g(z) \Psi_0(z) \) are bounded. Letting \( k > 0 \) be some number for which \( \min_{z \in [\hat{z}, \bar{z}]} g(z)(\Phi_0(z) + \kappa \Psi_0(z)) \geq -k \min_{z \in [\hat{z}, \bar{z}]} g(z)(\Phi_1(z) + \kappa \Psi_1(z)) \), we now consider a tax changes of the form \( \Delta(z; \hat{z}, r, k) \) that is in \( \Delta \), is zero outside of the interval \( B_{\epsilon}(\hat{z}) \), and satisfies \( \int_{\hat{z}}^{\hat{z} + r} \Delta(z; \hat{z}, r, k)^2 dz > k \int_{\hat{z}}^{\hat{z} + r} \Delta(z; \hat{z}, r, k)^2 dz \). Here the idea is to take \( \Delta(z; \hat{z}, r, k) \) to be a sufficiently narrow “bump” centered at \( z \); we provide an explicit example in Appendix E.6.

To complete the proof, consider \( \Delta \in \Delta \) defined by \( \Delta(z) \equiv \alpha_1 \Delta_-(z) + \alpha_2 \Delta_+(z) \) for \( \Delta_-(z) \equiv \Delta(z; \hat{z} - \frac{\epsilon}{2}, \frac{\epsilon}{2}, \frac{1}{2}, \frac{1}{2}, k), \Delta_+(z) \equiv \Delta(z; \hat{z} - \frac{\epsilon}{2}, \frac{\epsilon}{2}, \frac{1}{2}, \frac{1}{2}, k), \) and some constants \( \alpha_-, \alpha_+ \in \mathbb{R}^+ \). By choosing \( \alpha_- \) and \( \alpha_+ \) appropriately—and without setting both to zero—we can use ensure that

\[
D_{\Delta} H(0) = \alpha_- \int_{\text{supp} g} g(z) \psi(z) \Delta_-(z) dz + \alpha_+ \int_{\text{supp} g} g(z) \psi(z) \Delta_+(z) dz = 0. \tag{71}
\]

The planner’s second order condition then implies \( D^{2}_{\Delta} F(0) + \kappa D^{2}_{\Delta} H(0) \leq 0 \). However, applying Lemmas 6 and 7 to our construction implies the opposite, a contradiction:

\[
D^{2}_{\Delta} F(0) + \kappa D^{2}_{\Delta} H(0) = \alpha_- \int_{\text{supp} g} g(z) [\Phi_0(z) + \kappa \Psi_0(z)] \Delta_-(z)^2 + \alpha_+ \int_{\text{supp} g} g(z) [\Phi_0(z) + \kappa \Psi_0(z)] \Delta_+(z)^2 dz
\]

\[
+ \alpha_+ \int_{\text{supp} g} g(z) [\Phi_1(z) + \kappa \Psi_1(z)] \Delta_+(z)^2 dz + \alpha_+ \int_{\text{supp} g} g(z) [\Phi_1(z) + \kappa \Psi_1(z)] \Delta_+(z)^2 dz
\]

\[
\geq \alpha_- \int_{\text{supp} g} \left( \min_{z \in [\hat{z}, \bar{z}]} g(z)(\Phi_0(z) + \kappa \Psi_0(z)) \right) \Delta_-(z)^2 dz + \left( \min_{z \in [\hat{z}, \bar{z}]} g(z)(\Phi_1(z) + \kappa \Psi_1(z)) \right) \int_{\text{supp} g} \Delta_-(z)^2 dz
\]

\[
+ \alpha_+ \int_{\text{supp} g} \left( \min_{z \in [\hat{z}, \bar{z}]} g(z)(\Phi_0(z) + \kappa \Psi_0(z)) \right) \Delta_+(z)^2 dz + \left( \min_{z \in [\hat{z}, \bar{z}]} g(z)(\Phi_1(z) + \kappa \Psi_1(z)) \right) \int_{\text{supp} g} \Delta_+(z)^2 dz
\]

\[
> 0. \tag{72}
\]

Above, the first equality uses that \( \Delta_-(z) \) and \( \Delta_+(z) \) are never non-zero at any common \( z \). The final inequality uses that \( \alpha_- \) or \( \alpha_+ \) is non-zero and that \( \int_{\text{supp} g} \Delta_-(z)^2, \int_{\text{supp} g} \Delta_+(z)^2 > 0 \).

Thus, by computing the first and second derivatives of revenues and welfare, combining those expressions, and applying them to “narrow” deviations, we have shown that

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80 Any such tax change is in \( \Delta \) since by Lemma 1 is a Banach space.
that for

Footnote 29 in the statement of the Theorem.

marginal welfare weights to an exact welfare function.

within the income level. In a later step of the proof, we will connect these inferred
marginal welfare weight at each income level and then assign it to particular households
Part 1: Inferring / assigning welfare weights and curvatures

The proof has three main parts. The first main part is to infer from the tax schedule
a profile of marginal welfare weight at each income level and then assign it to particular households
within the income level. In a later step of the proof, we will connect these inferred
marginal welfare weights and curvatures and to demonstrate its various properties. The
third main part studies the planner’s Lagrangian given the social objective of the previous
part, and in particular makes a Lagrangian sufficiency argument for local optimality.

Part 1: Inferring / assigning welfare weights and curvatures

Implied marginal welfare weights

Our first step in constructing a welfare function will be to infer an implied average
marginal welfare weight at each income level and then assign it to particular households
in the statement of the proof\footnote{Footnote 29 in the statement of the Theorem.} and
where in the statement of the theorem,

Footnote 29 in the statement of the Theorem.

By the definition of $\Pi_{ABC}(z)$ and the assumptions in the statement of the theorem,
we have $\dot{\lambda}(z) > 0$ for all $z \in \text{supp} g$. In Appendices\footnote{Appendices E.7.1 and E.7.2} we establish several
other properties of $\dot{\lambda}(z)$ and $\dot{\lambda}^h$ related to measurability, continuity, integrability, and
conditional expectations. For the purpose of the main argument of the proof, only two
are essential: We show that (a) $\dot{\lambda}(z)$ is continuously differentiable on supp $g$ and (b) letting
$\eta_{\leq}(z; \epsilon) \equiv \mathbb{E}[\eta^h(R) \mid z^h(R) = z, \epsilon^h(R) < \epsilon], \epsilon_{\leq}(z; \epsilon) \equiv \mathbb{E}[\epsilon^h(R) \mid z^h(R) = z, \epsilon^h(R) < \epsilon],$
and $(\frac{\partial^2}{\partial z^2})_{\leq}(z; \epsilon) \equiv \mathbb{E}[\frac{\eta^h(R)^2}{\epsilon^h(R)} \mid z^h(R) = z, \epsilon^h(R) < \epsilon]$ be defined as in Assumption 6, we have
that for $x^h = 1, \eta^h(R), \epsilon^h(R),$ and $\frac{\eta^h(R)^2}{\epsilon^h(R)}$, $R(z_h^0)\dot{\lambda}(z_h^0)x_{\leq}(z_h^0; \epsilon)$ is a conditional expectation
for $R(z_h^0)\dot{\lambda}^h x^h$ conditional on $z_h^0$

Implied sufficient welfare curvature

We now proceed to use $\dot{\lambda}^h$ and our observations from the previous step of the proof
in order define a convenient curvature-of-welfare variable $\dot{z}^h$.

Recalling $\Psi_0(z)$ from Lemma 3, we begin by defining, for all $z \in \mathbb{R}_{\geq 0}$,

$$(\dot{\lambda}^h)(z) = \begin{cases} -1 - \frac{z}{\psi(z)} - \frac{\eta_{\leq}(z; \epsilon)}{R(z)} - \frac{d \log R(z)}{d \log z} \dot{\lambda}(z) \left( \frac{\psi}{\psi(z)} \right)_{\leq} (z; \epsilon) \\ -\frac{1}{2} \alpha(z) + \frac{d \log R(z)}{d \log z} \dot{\lambda}(z) \eta_{\leq}(z; \epsilon) + \frac{1}{2} \frac{d}{d \log z} \left( \dot{\lambda}(z) \eta_{\leq}(z; \epsilon) \right) \end{cases}, \text{ if } z \in \text{supp } g, \quad (74)$$

$\footnote{Our motivation for this particular choice of $\epsilon$ will become evident in the final step of the proof.}$
Next, we define a household-level version of this curvature, constructed so as to (a) aggregate up to the income conditional mean \( \hat{\gamma}(z) \) and (b) allocation all weight to households with low elasticities, as with \( \hat{\lambda} \).

\[
\hat{\gamma}^h = \begin{cases} 
\frac{(\hat{\lambda}(z))^{h}}{\hat{\lambda}^{h}P(z \in g)} & \text{if } z \in \text{supp } g \text{ and } \epsilon^h(R) \leq \epsilon \\
0, & \text{otherwise.}
\end{cases}
\] (75)

In Appendix [E.7.3] we establish several properties of \( (\hat{\lambda}^{h})(z) \) and \( \hat{\gamma}^h \) related to measurability, continuity, integrability, and conditional expectations. For the purpose of the main argument of the proof, only one is essential: We show that \( R(z^{h}_0)(\hat{\lambda}^{h})(z^{h}_0) \) is a conditional expectation for \( R(z^{h}_0)^{h}\hat{\lambda}^{h}\hat{\gamma}^{h} \) given \( z^{h}_0 \).

**Part 2: Definition and properties of a social objective**

At this point, we are finally ready to define a social objective using our inferred marginal welfare weights and curvatures \( \hat{\lambda} \) and \( \hat{\gamma}^h \). To do so, take \( \delta > 0 \) small enough that Assumption \( \delta \) applies, \( z^{h}(R) \) and \( c^{h}(R) \) exist and have two continuous and integrable Frechet derivatives on \( R \in R + B_{\delta}(0) \) (see Lemmas 2 and 3), and \( \hat{c}^h(u) \equiv u^h(\cdot, z^h_0)^{-1}(u) \) is defined for all \( u \in V^h(R + B_{\delta}(0)) \) (see Appendix [E.7.4]). Then define \(((w^h)_{h \in \mathcal{H}}, G) \) by:

\[
w^h : \text{Im}(w^h) \to \mathbb{R} \cup \{ -\infty \} \quad u \mapsto \begin{cases} 
\hat{\lambda}^h \int_{[c^h_0, u]} e^{\Phi(\hat{\gamma}^h(\log \hat{c}^h(\bar{u}) - \log c^h_0))} u^h(\hat{c}^h(\bar{u}), z^h_0) d\bar{u}, & \text{if } u \in V^h(R + B_{\delta}(0)) \\
0, & \text{else}
\end{cases}
\] (76)

\[G = \int (z^h_0 - c^h_0) d\mu\]

where \( \Phi(\cdot) \equiv \sqrt{2\pi} (\Phi_0(\cdot) - \frac{1}{2}) \) and \( \Phi_0 \) is the standard normal CDF.

82 The only properties of \( \Phi \) we will use are that \( \Phi \) is infinitely continuously differentiable, \( \Phi(0) = 0, \Phi'(0) = 1, \) and some scalar \( B \) bounds \( |\Phi(x)| \) and \( |\Phi'(x)| \) across all \( x \in \mathbb{R} \).
and $R(z_h^b)\tilde{\lambda}(z_h^b)\left(\frac{\eta_2^c}{\epsilon}\right) \leq (z_h^b; \epsilon)$, and $R(z_0^h)(\tilde{\lambda}\gamma)(z_0^h)$ respectively.

**Part 3: Lagrangian sufficiency**

**Derivatives of planner’s Lagrangian**

Define $\mathcal{L} : \mathcal{R} + B_3(0) \to \mathbb{R}$ by $\mathcal{L}(\tilde{R}) \equiv \int w^h \circ V^h(\tilde{R})d\mu + \int \left(z^h(\tilde{R}) - c^h(\tilde{R})\right)d\mu - G$.

Toward the eventual goal of making a Lagrangian sufficiency argument, we now consider and compute the derivatives of $\mathcal{L}(\tilde{R})$. By Lemmas 3 and 5, we may take $\delta$ small enough that, by the linearity of differentiation, $\mathcal{L}(\tilde{R})$ is twice-continuously Frechet-differentiable on $\mathcal{R} + B_3(0)$. In Appendix E,7.6 we show that—under Assumption 6—strengthened versions of Lemmas 6 and 7 hold for the objective defined above, so that for all $\Delta \in \Delta$,

$$D_\Delta \mathcal{L}(R) = \int_{\supp g} g(z) [\lambda(z) + \psi(z)] \Delta(z)dz$$

$$D_{\Delta \Delta} \mathcal{L}(R) = \int_{\supp g} g(z) \left[\left(\Phi_0(z) + \Psi_0(z)\right) \left(\frac{\Delta(z)}{R(z)}\right)^2 + \left(\Phi_1(z) + \Psi_1(z)\right) \left(\frac{\Delta(z)}{R(z)}\right)^2\right]dz,$$

where $\psi(z), \Phi_0(z), \Psi_0(z), \Phi_1(z), \Psi_1(z)$ are as defined in Lemmas 6 and 7, and recall these functions are continuous on $z \in \supp g$.

This expression for the Lagrangian’s first derivative has a straightforward implication: By (73) and (77), $\lambda(z) + \psi(z) = 0$, so $D\mathcal{L}(R) = 0$. Turning now to the Lagrangian’s second derivative, we claim there exists a scalar $b > 0$ such that for all $z \in \supp g$,

$$\Phi_0(z) + \Psi_0(z) \leq -b(R(z) + z) \quad \text{and} \quad \Phi_1(z) + \Psi_1(z) \leq -b(R(z) + z). \quad (79)$$

To see this, first note that (a) by the definitions of $(\tilde{\lambda}\gamma)(z)$ (see (74)) and $\Phi_0(z)$, and (b) since—as $\lambda^h(R) = \tilde{\lambda}^h$ and $(\lambda\gamma)^h(R) = \tilde{\lambda}\gamma^h$ for measure one of households—the conditional expectations discussed above remain valid if $\tilde{\lambda}^h$ and $\tilde{\lambda}\gamma^h$ are replaced by $\lambda^h(R)$ and $(\lambda\gamma)^h(R)$, respectively, we have that for all $z \in \supp g$,

$$\Phi_0(z) = R(z) \left[(\lambda\gamma)(z) + \frac{d\log R(z)}{d\log z} \lambda(z) \left(\frac{\eta_2}{\epsilon}\right)(z; \epsilon) + \frac{1}{2} \left(\alpha(z) + \frac{d\log R(z)}{d\log z}\right) \lambda(z) \eta(z; \epsilon) - \frac{1}{2} \frac{d(\tilde{\lambda}\gamma)(z; \epsilon)}{d\log z}\right]$$

$$\implies \Phi_0(z) + \Psi_0(z) = R(z) \left[-1 - \frac{z}{R(z)} - \frac{\Psi_0(z)}{R(z)}\right] + \Psi_0(z) = -R(z) - z. \quad (80)$$

So $\Phi_0(z) + \Psi_0(z) \leq -b(R(z) + z)$ for any $b \in (0, 1)$. Moreover, by the definitions of $\Phi_1(z)$ and $\Psi_1(z)$, we have, for all $z \in \supp g$,

$$\Phi_1(z) + \Psi_1(z) = R(z) \frac{d\log R(z)}{d\log z} \frac{(\lambda\gamma)(z)}{R(z)} - \left(\Pi_{DEFG}(z) - B^R(\tilde{\lambda}_c, R(z) + \tilde{\lambda}_z, z) - \tilde{b}_c R(z) - \tilde{b}_z\right). \quad (81)$$

Above, the second inequality uses Assumption 11 the definitions of $\tilde{\lambda}_c$ and $\tilde{\lambda}_z$, and the bounding assumption on $\Pi_{DEFG}(z)$ in the statement of the theorem. Recalling our definition of $\epsilon < \frac{1}{4\pi} \min \left[\frac{\tilde{\lambda}_c}{\tilde{\lambda}_z}, \frac{\tilde{\lambda}_z}{\tilde{\lambda}_z}\right]$, we have shown that there exists $b > 0$ such that $\Phi_1(z) + \Psi(z) \leq -b(R(z) + z)$.

We summarize our observations about $\mathcal{L}$’s derivatives as follows: $\mathcal{L}$ is twice-continuously
differentiable on \( R + B_\delta(0) \) and there exists \( b > 0 \) such that for all \( \Delta \in \Delta \),
\[
D_\Delta \mathcal{L}(R) = 0 \quad \text{and} \quad D^2_{\Delta \Delta} \mathcal{L}(R) \leq -b||\Delta||^2
\]
where
\[
||\Delta||_* \equiv \left[ \int g(z)(z + R(z)) \left( \left( \frac{\Delta(z)}{R(z)} \right)^2 + \left( \frac{\Delta'(z)}{R'(z)} \right)^2 \right) dz \right]^{\frac{1}{2}}
\]  
\[83\]

### Lagrangian sufficiency on a restricted domain

To finish the proof, it suffices to show that, given any \( M > 0 \), there exists sufficiently small \( \delta > 0 \) so that \( \mathcal{L}(\tilde{R}) \) is maximized within \( R + B_\delta(0) \cap \Delta^*_M \) by \( \tilde{R} = R \), where \( \Delta^*_M = \{ \Delta \in \Delta \mid ||\Delta|| \leq M||\Delta||_* \} \) is as defined in the theorem statement. A Lagrangian sufficiency argument then completes the proof. The reason we restrict the domain to \( \Delta^*_M \) is that this domain rules out sequences of variations which become much larger in the sense of \( ||\cdot|| \) (the relevant norm for Taylor’s theorem) than in the sense of \( ||\cdot||_* \) (the relevant sense for our bounds on the Lagrangian’s second derivative).

To see this, first note that for any \( R + \Delta \in R + B_\delta(0) \), Taylor’s theorem applied along the line between \( R \) and \( \tilde{R} \) (allowed by \( \mathcal{L} \)’s twice-continuous differentiability), we have
\[
\mathcal{L}(R + \Delta) = \mathcal{L}(R) + D_\Delta \mathcal{L}(R) + \frac{1}{2} D^2_{\Delta \Delta} \mathcal{L}(R + \alpha_\Delta \Delta)
\]
\[83\]
where
\[
f or \text{some } \alpha_\Delta \in [0, 1]. \text{ We next argue that } \frac{D^2_{\Delta \Delta} \mathcal{L}(R + \alpha_\Delta \Delta) - D^2_{\Delta \Delta} \mathcal{L}(R)}{||\Delta||^2} \text{ converges to 0 uniformly across } \Delta \text{ and } \alpha_\Delta \text{ for all sequences } \Delta \to 0. \text{ To see this, note that by } \mathcal{L} \text{'s twice-continuous differentiability, we have that for all } \delta > 0 \text{ such that for all } \tilde{\Delta} \in B_\delta(0) \text{ and all non-zero } \hat{\Delta}, \tilde{\Delta} \in \Delta,
\[
\left| \frac{D^2 \mathcal{L}_{\tilde{\Delta} \hat{\Delta}}(R + \tilde{\Delta}) - D^2 \mathcal{L}_{\tilde{\Delta} \hat{\Delta}}(R)}{||\Delta|| ||\hat{\Delta}||} \right| < \delta.
\]
\[84\]
The desired conclusion is implied by taking \( \tilde{\Delta} = \alpha_\Delta \Delta \) and \( \hat{\Delta} = \hat{\Delta} = \Delta \). In particular, this implies that the last term in \[83\] is \( o(||\Delta||^2) \) in the sense that
\[
\frac{1}{2} \left( D^2_{\Delta \Delta} \mathcal{L}(R + \alpha_\Delta \Delta) - D^2_{\Delta \Delta} \mathcal{L}(R) \right) \to 0 \quad \text{as} \quad ||\Delta|| \to 0.
\]
\[85\]
Combining these observations with the previous step, we may, for any \( R + \Delta \in R + B_\delta(0) \), write
\[
\mathcal{L}(R + \Delta) = \mathcal{L}(R) + D_\Delta \mathcal{L}(R) + \frac{1}{2} D^2_{\Delta \Delta} \mathcal{L}(R) + o(||\Delta||^2)
\]
\[
\leq \mathcal{L}(R) - b \frac{||\Delta||^2}{2} + o(||\Delta||^2)
\]
\[
\implies \mathcal{L}(R + \Delta) - \left( \mathcal{L}(R) - \frac{b}{2} \frac{||\Delta||^2}{2} \right) \leq - \frac{b}{2} \frac{||\Delta||^2}{2} + o(||\Delta||^2)
\]

\[86\]

---

\[84\] Put simply, Lagrangian sufficiency says that if there exists a Lagrange multiplier—in our case 1—so that a feasible point—in our case \( \tilde{R} \)—maximizes the Lagrangian among all points in some set that includes all feasible points, then that point solves the constrained optimization problem.
Finally, we claim that—for any $M > 0$—there exists $\tilde{\delta}$ such that for all $R + \Delta \in R + B_\delta(0) \cap \Delta_M^*$, $\mathcal{L}(R + \Delta) \leq \mathcal{L}(R) - \frac{b}{2} \|\Delta_n\|^2$; note this implies $\mathcal{L}(R + \Delta) \leq \mathcal{L}(R)$.

To see this, suppose otherwise, i.e. there exists a sequence $\Delta_n \to 0$, with each $\Delta_n \in B_\delta(0) \cap \Delta_M^*$, such that for all $n$, $\mathcal{L}(R + \Delta_n) > \mathcal{L}(R) - \frac{b}{2} \|\Delta_n\|^2$. By (86), this implies

$$0 < -\frac{b}{2} \frac{\|\Delta_n\|^2}{2} + o(\|\Delta\|^2) \leq -\frac{b}{2\sqrt{M}} \frac{\|\Delta_n\|^2}{2} + o(\|\Delta_n\|^2).$$ (87)

where the second inequality is by the definition of $\Delta_M^*$. For small enough $n$, the RHS is strictly negative, a contradiction.

This guarantees that—within $\Delta_M^*$—the Lagrangian of the planner we have constructed is locally maximized at $R$. By Lagrangian sufficiency, this completes the proof.

### C.3 Proof of claims from “sort and extort” section (4.3.1)

Suppose that each household $h$ belongs to one of finitely many groups $i^h \in I$. We denote income-conditional statistics $x^h|z_0^h = z, i^h = i$ within each group by $x(z; i)$ and so on. To begin, we will make use of the following simple Lemma:

**Lemma C.1.** For any differentiable function $f(z, i)$,

$$\left(\alpha(z) - \frac{d}{d\log z}\right) \mathbb{E}[f(z; i^h) \mid z_0^h = z] = \mathbb{E}\left[\left(\alpha(z; i^h) - \frac{d}{d\log z}\right) f(z; i^h) \mid z_0^h = z\right]$$ (88)

**Proof.** Recalling that $\alpha(z) = \frac{d}{d\log z} \log z + \frac{d}{d\log z} f(z)$ and multiplying through by $-h(z)$, note that

$$-g(z) \left(\alpha(z) - \frac{d}{d\log z}\right) \mathbb{E}[f(z; i^h) \mid z_0^h = z] = g(z) \frac{d}{d\log z} \mathbb{E}[f(z; i^h) \mid z_0^h = z] + g(z) \frac{d}{d\log z} \mathbb{E}[f(z; i^h) \mid z_0^h = z]$$

$$= \frac{d}{dz} (g(z)z) \mathbb{E}[f(z; i^h) \mid z_0^h = z] + g(z) \frac{d}{d\log z} \mathbb{E}[f(z; i^h) \mid z_0^h = z]$$

$$= \frac{d}{dz} (g(z)z) \mathbb{E}[f(z; i^h) \mid z_0^h = z] + g(z) \frac{d}{d\log z} \mathbb{E}[f(z; i^h) \mid z_0^h = z]$$

$$= \frac{d}{dz} \left(\sum_{i \in I} g(z; i)z f(z; i)\right)$$

$$= -\sum_{i \in I} g(z; i) \left(\alpha(z; i) - \frac{d}{d\log z}\right) f(z; i)$$

$$= -g(z) \mathbb{E}\left[\left(\alpha(z; i^h) - \frac{d}{d\log z}\right) f(z; i^h) \mid z_0^h = z\right]$$

Applying this Lemma to $\text{DEFG}$ and substituting for $\varepsilon^+(z)$ implies that $\text{DEFG}$ can be
written as:

\[
0 \geq - (1 + R'(z)) \mathbb{E}[\varepsilon(z; i^h) \mid z_0^h = z] + (1 - R'(z)) \left( \mathbb{E} \left[ \left( \alpha(z; i^h) - \frac{d}{d \log z} \right) \varepsilon^2(z; i^h) \mid z_0^h = z \right] + \mathbb{E} \left[ \varepsilon(z; i^h) \frac{d}{d \log z} \varepsilon(z; i^h) \mid z_0^h = z \right] \right)
\]

\[
= - \mathbb{E} \left[ (1 + R'(z)) \varepsilon(z; i^h) + (1 - R'(z)) \left( \left( \alpha(z; i^h) - \frac{d}{d \log z} \right) \varepsilon^2(z; i^h) + \varepsilon(z; i^h) \frac{d}{d \log z} \varepsilon(z; i^h) \right) \mid z_0^h = z \right]
\]

\[
= \sum_{i \in I} \mathbb{P} [i^h = i \mid z_0^h = z] \Pi_{DEFG}(z)
\]

(90)

Next, we assume that within each group \(i\), preferences satisfy the function form studied in \textbf{Werning [2007]}:

\[
u^h(c, z) = v^i_c(c) - v^i_z(z) / \theta^h,
\]

(91)

where \(v^i_c\) and \(v^i_z\) are increasing and \(v^i_c\) is concave and \(v^i_z\) is convex.

\textbf{Proposition 1.} If preferences with a group \(i \in I\) satisfy [91], then \(\Pi_{ABC}(z) \implies \Pi_{DEFG}(z)\)

\textbf{Proof.} Starting from \textbf{(DEFG)}, we divide out by a factor of \(z\varepsilon(z; i)\). Letting \(h(z, i)\) be the type (if one exists) in group \(i\) who supplies labor \(z\), we this gives us

\[
0 \geq - (1 + R'(z)) + (1 - R'(z)) \left[ \alpha(z) \varepsilon(z; i) - \frac{d\varepsilon(z; i)}{d \log z} + \frac{d}{d \log z} \varepsilon(z; i) \right]
\]

where here the fact that the last two terms cancel is an easily-verifiable property of [91].

Next we rearrange the first-order condition \textbf{(ABC)} to get

\[
(1 - R'(z)) (\alpha(z; i) \varepsilon(z; i) - \varepsilon'(z; i) z) = R'(z) (1 + \Pi_{ABC}(z)) - (1 - R'(z)) \frac{d \log R(z)}{d \log z} \eta(z; i) - \frac{d \log R'(z)}{d \log z} \varepsilon(z; i)
\]

(93)

Combining the two previous equations—the former from the second-order condition and the latter from the first-order condition—\textbf{(DEFG)} for group \(i\) is equivalent to:

\[
0 \geq \Pi_{ABC}(z) - 1 - (1 - R'(z)) \frac{d \log R(z)}{d \log z} \eta(z; i) - \frac{d \log R'(z)}{d \log z} \varepsilon(z; i)
\]

(94)

In order to simplify the LHS of this expression, we note that—from the definition of income and compensated elasticities in [121]—

\[
1 + (1 - R'(z)) \frac{d \log R(z_0^h)}{d \log z_0^h} \eta^h(R) + \frac{d \log R'(z_0^h)}{d \log z_0^h} \varepsilon^h(R)
\]

\[
= \frac{d \log M^h(c_0^h, z_0^h)}{d \log c_0^h} \frac{d \log R(z_0^h)}{d \log z_0^h} + \frac{d \log M^h(c_0^h, z_0^h)}{d \log z_0^h} - \frac{d \log R'(z_0^h)}{d \log z_0^h} - (I - R'(z_0^h)) \frac{d \log R(z_0^h)}{d \log z_0^h} \frac{d \log M^h(c_0^h, z_0^h)}{d \log c_0^h} + \frac{d \log R'(z_0^h)}{d \log z_0^h}
\]

\[
= \varepsilon^h(R) \left( \frac{d \log M^h(c_0^h, z_0^h)}{d \log z_0^h} + R'(z) \frac{d \log R(z_0^h)}{d \log z_0^h} \frac{d \log M^h(c_0^h, z_0^h)}{d \log c_0^h} \right)
\]

(95)
So \((DEFG)\) is equivalent to
\[
\Pi_{ABC}^i(z) \leq \frac{1}{R'(z)} \varepsilon(z; i) \left( \frac{\partial \log M(R(z), z; i)}{\partial z} + R'(z) \frac{d \log R(z)}{d \log z} \frac{\partial \log M(R(z), z; i)}{\partial c} \right).
\]

Since \(R'(z) > 0\) by the proof of Lemma 2, since compensated elasticities are positive, and since the concavity and convexity of \(v^e_i\) and \(v^z_i\), respectively, imply \(M^h(c, z)\) is increasing in consumption and labor for all \(h\), the RHS is positive. \((ABC)\) for group \(i \in I\) therefore implies \((DEFG)\) for group \(i\).

\(\square\)

### D Empirical Appendix

#### D.1 Selection of bandwidths for local regressions

For picking our bandwidths, we minimize the leave-one-one cross validations criteria, which is the average of squared leave-one-out residuals, that is
\[
LOOCV = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{g}_{-i}(x_i))^2;
\]
where \(\hat{g}_{-i}(x_i)\) is the predicted value for \(y_i\) using the estimated model \(\hat{g}_{-i}\) that does use the observation \(i\), but evaluated at the covariate values \(x_i\). The average of these residuals is an estimate for the asymptotic mean integrated square error (AMISE) of model with bandwidth \(h\). By minimizing it, we pick the model with lowest estimated AMISE [Li and Racine, 2007].

For regressions with a large number of observations, the procedure above is computationally demanding. To speed up computations we use the procedure described in Racine [1993], where the leave-one-out cross validation is computed in subsamples and then is scaled down, using the fact that the optimal bandwidth should be proportional to \(c \sigma_x n^{-1/3}\), where \(c\) is a constant that does not depend on the number of observations \(n\), and \(\sigma_x\) is the standard deviation of \(x\).

#### D.2 Non-parametric identification of elasticity moments

In this section we show that we can recover all moments of the distribution of elasticities from moments in the data. We assume that behavioral responses of each worker to tax changes are linear, that is, each of them has constant elasticities.

Under this assumption, the change in income and the change in marginal retention are related through an equation of the form:
\[
y_t^h = a_t^h + b_t^h x_t^h
\]
We now show that we can recover all moments of the joint distribution of \((a_t^h, b_t^h)\).

We now raise the structural equation above to the \(n - power\) and algebraically manipulate it into a regression equation:
\[
(y_t^h)^n = (a_t^h + b_t^h x_t^h)^n
\]
\[(y^h_t)^n = \sum_{k=0}^{n} \binom{n}{k} (a^h_t)^{n-k} (b^h_t)^k \]
\[(y^h_t)^n = \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}[(a^h_t)^{n-k}(b^h_t)^k](x^h_t)^k + \sum_{k=0}^{n} \binom{n}{k} \left( (a^h_t)^{n-k}(b^h_t) - \mathbb{E}[(a^h_t)^{n-k}(b^h_t)^k] \right) (x^h_t)^k \]

Now, notice that, assuming \(x^h_t\) is randomly assigned, for any \(k\) and \(k'\),
\[\mathbb{E} \left[ (x^h_t)^{k'k} \left( (a^h_t)^{n-k}(b^h_t) - \mathbb{E}[(a^h_t)^{n-k}(b^h_t)^k] \right) \right] = 0.\]

Therefore, the equation above is a regression equation, where the coefficients are moments of the distribution of \(a^h_t\) and \(b^h_t\). Crucially, note that all moments of the joint distribution of \((a^h_t, b^h_t)\) take the form \(\mathbb{E}[(a^h_t)^{n-k}(a^h_t)^k]\) for some \(n\) and \(k\). So we can recover it from some such regression, assuming that there is enough variation in \(x\) (which prevents collinearity).

### D.3 Derivation of mechanism regression equation

In this section, we show that the regression equation 22 recovers the variance of elasticities within brackets.

We start from the following expression, based on (21):
\[\mathbb{E}[\varepsilon^h | z, \log R^\prime_t] \approx \mathbb{E}[\varepsilon^h | z, \log R^\prime] + \alpha_t(z) \mathbb{Var}[\varepsilon^h | z] (\log R^\prime_t(z) - \log R^\prime(z)).\]

Writing the elasticity \(\varepsilon^h\) as the sum of its expectation and an expectational error \(\xi^h\), we use this expression to substitute in for elasticities in the definition of the elasticity elasticity as the approximate change in income that results from an exogenous change in marginal taxes at the initial income level:
\[\Delta \log z^h_t \approx \varepsilon^h \Delta \log R^\prime_t(z^h_t) + a^h_t\]
\[\approx \left[ \mathbb{E}[\varepsilon^h | z^h, \log R^\prime] + \alpha(z) \mathbb{Var}[\varepsilon^h | z^h] (\log R^\prime_t(z^h_t) - \log R^\prime(z^h_t)) + \xi^h_t \right] \Delta \log R^\prime_t(z^h_t) + a^h_t\]
where here the expectation and variances are over the household \(\tilde{h}\), whose income we condition on being equal to \(z^h\).

Finally, we add and subtract terms to arrive at a regression equation:
\[\Delta \log z^h_t \approx \mathbb{E}[\mathbb{E}[\varepsilon^h | z^h, \log R^\prime]] \cdot \Delta \log R^\prime(z^h)\]
\[+ \mathbb{E}[\mathbb{Var}[\varepsilon^h | z^h]] \cdot \alpha_t(z^h) (\log R^\prime_t(z^h_t) - \log R^\prime(z^h_t)) \Delta \log R^\prime_t(z^h)\]
\[+ \mathbb{Var}[\varepsilon^h | z^h] - \mathbb{E}[\mathbb{Var}[\varepsilon^h | z^h]] \alpha_t(z^h_t) \mathbb{E}[\mathbb{Var}[\varepsilon^h | z^h] \log R^\prime(z^h_t)] \Delta \log R^\prime_t(z^h_t)\]
\[+ \mathbb{E}[\varepsilon^h | z^h, \log R^\prime] - \mathbb{E}[\mathbb{E}[\varepsilon^h | z^h, \log R^\prime]] \Delta \log R^\prime_t(z^h_t)\]
\[+ \xi^h_t \Delta \log R^\prime_t(z^h_t) + a^h_t,\]
where all nested expectations are over \(\tilde{h}\) for the inner expectation and \(\hat{h}\) for the outer expectation.

This is a regression equation, and it is identified under the assumption that tax changes are randomly assigned. The coefficient on \(\alpha_t(z^h_t)\)(\log R^\prime_t(z^h_t) - \log R^\prime(z^h_t))\Delta \log R^\prime_t(z^h_t)\) recovers the average variance measure \(\mathbb{E}[\mathbb{Var}[\varepsilon^h | z^h]]\).
E  Omitted Proofs

This appendix contains proofs of supporting results used in the main theorems, as well as various technical details.

E.1 Proof of Lemma 1

This proof shows that $(\Delta, ||\cdot||)$ is a Banach space.

$\Delta$ is a real vector space by standard arguments (the main property to consider is closure under addition). Moreover, $||\cdot||$ induces a norm on $\Delta$; this is easy to check:

- If $||\Delta|| = 0$, then by definition, $\Delta(z) = 0$ for all $z > 0$. Continuity then implies $\Delta(z) = 0$ for all $z \geq 0$. In the other direction, if $\Delta(z) = 0$, then $||\Delta|| = 0$ by definition.

- For any $a \in \mathbb{R}$, $||a\Delta|| = ||a\cdot||\Delta||$.

- To show the triangle inequality, take $\Delta, \tilde{\Delta} \in \Delta$. Let $B^\Delta$ and $B^\tilde{\Delta}$ be bounds for which

$$\forall z \in \mathbb{R}_{>0}, |\Delta(z)| \leq B^\Delta |R(z)| \text{ and } |\Delta'(z)|, \left| \frac{d\Delta'(z)}{d \log z} \right|, \left| \frac{d^2 \Delta'(z)}{d \log z^2} \right| \leq B^\Delta |R'(z)|$$

and

$$\text{and } |\tilde{\Delta}(z)| \leq B^{\tilde{\Delta}} |R(z)| \text{ and } |\tilde{\Delta}'(z)|, \left| \frac{d\tilde{\Delta}'(z)}{d \log z} \right|, \left| \frac{d^2 \tilde{\Delta}'(z)}{d \log z^2} \right| \leq B^{\tilde{\Delta}} |R'(z)|$$

By the linearity of differentiation, we have that

$$\forall z \in \mathbb{R}_{>0}, |\Delta(z) + \tilde{\Delta}(z)| \leq (B^\Delta + B^{\tilde{\Delta}})|R(z)|$$

and

$$|\Delta'(z) + \tilde{\Delta}'(z)|, \left| \frac{d \left( \Delta'(z) + \tilde{\Delta}'(z) \right)}{d \log z} \right|, \left| \frac{d^2 \left( \Delta'(z) + \tilde{\Delta}'(z) \right)}{d \log z^2} \right| \leq (B^\Delta + B^{\tilde{\Delta}})|R'(z)|$$

which implies $\Delta + \tilde{\Delta} \in \Delta$.

In order to argue that $(\Delta, ||\cdot||)$ is a real Banach space, it remains to verify that it is complete. To this end, consider a sequence $(\Delta_n)_{n=0}^{\infty}$ of functions in $\Delta$. Moreover, suppose the sequence is Cauchy with respect to $||\cdot||$. We will show the sequence converges to some limit contained in $\Delta$.

To begin, note that since $\Delta_n$ is Cauchy, so are—for each $z \in \mathbb{R}_{>0} = \Delta_n(z), \Delta'_n(z), \frac{d\Delta'_n(z)}{d \log z}$, and $\frac{d^2 \Delta'_n(z)}{d \log z^2}$—each therefore converges pointwise to some functions $\Delta^0(z), \Delta^1(z), \Delta^2(z), \Delta^3(z) : \mathbb{R}_{>0} \to \mathbb{R}$. Note that the sequence $\Delta_n(0)$ is also Cauchy. Otherwise the continuity of the $\Delta_n$s implies that for some $\epsilon > 0$, there exist arbitrarily large $n$ and $m$ as well as $z \in (0, 1)$ for which $\Delta_n(z) - \Delta_m(z) > \epsilon$. Since $R(z)$ is, by continuity, bounded on $[0, 1]$, this would violate that $\Delta_n$ is Cauchy. So $\Delta^0(z)$ is defined on $\mathbb{R}_{>0}$.

We next establish two facts about these functions $\Delta^k(z)$: First, for each $k = 0, 1, 2, 3$ there exists $B_k \in \mathbb{R}$ such that

$$\forall z \in \mathbb{R}_{>0}, |\Delta^k(z)| \leq B_k |R^{(\max\{1,k\})}(z)|.$$  

To see this, suppose not. In this case, the bounds $B_n = ||\Delta||_n$ associated with each $\Delta_n$ must diverge. But it is easy to see this violates that the sequence is Cauchy. Second, we
note that each $\Delta^k(z)$ function is continuous in $z$. This follows from the fact that, for any $z \in \mathbb{R}_{>0}$, $|R^{(\max[1,k])}(z)|$ achieves some max on $[z/2, 2z]$ by continuity (from Assumption 1). The definition of $||\cdot||$ then implies that whichever of $\Delta_n(\cdot), \Delta'_n(\cdot), \frac{d\Delta_n(\cdot)}{d \log z},$ and $\frac{d^2\Delta_n(\cdot)}{d \log z^2}$ converges to $\Delta^k(\cdot)$ on $[z/2, 2z]$ does so uniformly. Since each $\Delta_n$ is three-times continuous differentiable and uniform convergence preserves continuity, this implies $\Delta^k$ is continuous on $[z/2, 2z]$; varying $z$, we get that $\Delta^k$ is continuous on $\mathbb{R}_{>0}$.

In the case of $k = 0$, both of these observations on $\Delta^{k=0}$ extend to $\mathbb{R}_{\geq 0}$. First, the fact that $\Delta_n \to \Delta^0$ uniformly on $(0, 1]$, the fact that $\Delta_n$ is continuous on $[0,1]$, and the triangle inequality imply that $\Delta_n \to \Delta^0$ uniformly on $[0,1]$. So $\Delta^0$ inherits $\Delta_n$’s continuity on $\mathbb{R}_{\geq 0}$. Applying this continuity and that of $R(z)$ to the fact that for all $z > 0$, $|\Delta^0(z)| \leq B_k|R(z)|$ implies this is also true at $z = 0$.

Finally, we claim that for all $z \in \mathbb{R}_{>0}$, $\Delta^0(z)$ exists and equals $\Delta^1(z), \frac{d\Delta^1(z)}{d \log z}$ exists and equals $\Delta^2(z), \frac{d\Delta^2(z)}{d \log z}$ exists and equals $\Delta^3(z).$ To see this, recall we have already argued that for all $m \in \mathbb{N}$, $\Delta_n \to \Delta^0, \Delta_n' \to \Delta^1, \frac{d\Delta_n(\cdot)}{d \log z} \to \Delta^2,$ and $\frac{d^2\Delta_n(\cdot)}{d \log z^2} \to \Delta^3$ uniformly on $[\frac{1}{m}, m]$. The lemma stated below therefore implies that each of the derivatives described above exists and coincides in the desired way on $[\frac{1}{m}, m].$ Applying this argument for each $m \in \mathbb{N}$ gives us the desired claim on $\mathbb{R}_{>0}$.

**Lemma E.1.** Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of differentiable functions whose derivatives $f'_n$ are continuous. If $f_n \to f$ uniformly and $f'_n \to g$ uniformly, then $f$ is differentiable and $f' = g.$

**Proof.** This is a standard fact in real analysis. See, e.g., Bakker.

At this point we have shown that the sequence $(\Delta_n)$ converges to a function $\Delta^0$ that is continuous on $\mathbb{R}_{\geq 0}$ and that has first, second, and third derivatives $\mathbb{R}_{>0}$ (this follows from the previous step because $z > 0$ on $\mathbb{R}_{>0}$). We have shown that these derivatives are continuous on $\mathbb{R}_{>0}$ (similarly). Finally, we have shown the existence of some $B = \max_{k=0,1,2,3} B_k$ so that $|\Delta^0(z)| \leq B|R(z)|$, $|\Delta^0'(z)| \leq B|R'(z)|$, $|\frac{d\Delta^0(\cdot)}{d \log z}| \leq B|R'(z)|,$ and $|\frac{d^2\Delta^0(z)}{d \log z^2}| \leq B|R'(z)|$. Together, these observations imply $\Delta^0 \in \Delta$.

### E.2 Proof of Lemma 2

We complete the proof in seven steps. First, we establish a convenient fact we will use throughout; second, we show the existence of a unique labor supply function within $R + B_\ell(0)$ for some $\ell$ (common across $h \in \mathcal{H}$); and, third and fourth, we show this labor supply function is twice continuously differentiable and measurable. Fifth and sixth, we provide explicit expressions for the first and second derivatives of labor supply; and, seventh, we show these as well are measurable. Eighth, we show that $R(z^h(\tilde{R}))$ inherits the relevant properties from $z^h(\tilde{R}).$

**Positivity of tax schedule**

Fix any $h \in \mathcal{H}$. We will show that an implication of the fact that $z^h_0 > 0$ is that for all $z > 0, R(z) > 0$ and $R'(z) > 0$. Under Assumption 3, this implies that for all $h \in \mathcal{H},$ $R(z^h_0), R'(z^h_0) > 0$.

We begin with the $R$ case and then proceed to the $R'$ case. To see the former, suppose not. Then since $R$ is continuous, there exists some highest $z < z^h_0$ for which $R(z) = 0.$
But then for all \( z' \in (z, z_0^h) \), we have

\[
\log R(z') = \int_{\log z_0^h}^{\log z'} \frac{R'(e^z) e^z}{R(e^z)} \, dz' \in [-B^R|\log z_0^h - \log z'|, B^R|\log z_0^h - \log z'|]
\]

(101)

\[\implies R(z') \in \left[ e^{-B^R|\log z_0^h - \log z'|}, e^{B^R|\log z_0^h - \log z'|} \right] \cdot R(z_0^h)\]

where the underbrace is by Assumption 1. Since \( R(z') \geq e^{-B^R|\log z_0^h - \log z'|} R(z_0^h) > 0 \) for all \( z' > z, R(z) > 0 \) by continuity, a contradiction.

We may apply a similar argument to \( R' \)’s first derivative, leveraging the observation that, since \( z_0^h > 0 \) and \( u^h \) and \( R \) are differentiable, \( h \)’s labor supply satisfies the first-order condition:

\[
u^h_{c}(c_0, z_0^h) R'(z_0^h) + u^h_{z}(c_0, z_0^h) = 0 \quad (102)
\]

where the underbraces are by Assumption 2 and the fact that \( z_0^h, c_0^h > 0 \). This implies \( R'(z_0^h) > 0 \). The same argument as above applied to \( R' \) instead of \( R \) the implies that \( R'(z) > 0 \) at all \( z > 0 \).

**Existence and uniqueness of labor supply function**

We now claim that, for any \( \Delta \in B_{\bar{c}(\bar{c})}(0) \), \( h \)'s problem \( \bar{c}(\bar{c}) \) at \( R + \Delta \) has a solution within \( e^{B_{\bar{c}(\bar{c})}(z(R))} \). To see this, first recall that by assumption, for any \( z \not\in e^{B_{\bar{c}(\bar{c})}(z(R))} \),

\[
u^h_1(R(z + \Delta(z), e^\bar{c}, z) \leq u^h(R(z_0^h)e^{-\bar{c}}, z_0^h) \quad (103)
\]

Second, we claim that for \( ||\Delta|| \leq \bar{c}(\bar{c}) \equiv \min_{\bar{c}(\bar{c})} \), we have \( R(z) + \Delta(z) \in [R(z)e^{-\bar{c}}, R(z)e^{\bar{c}}] \). To see this, note that \( ||\Delta|| < \bar{c}(\bar{c}) \leq \bar{c} < e^{\bar{c}} - 1 \) implies \( R(z) + \Delta(z) < R(z)e^{\bar{c}} \) for all \( z \in \mathbb{R}_{\geq 0} \). Moreover it is easy to verify that for all \( \bar{c} > 0 \), \( 1 - \frac{\min_{\bar{c}(\bar{c})}}{2} < e^{-\bar{c}}, \) implying that for \( ||\Delta|| < \bar{c}(\bar{c}), R(z) + \Delta(z) \geq R(z)e^{-\bar{c}} \) with strict inequality wherever \( |R(z)| > 0 \).

Combining these observations and using that utility is strictly increasing in consumption, we have for any \( z \not\in e^{B_{\bar{c}(\bar{c})}(z(R))} \) that

\[
u^h_1(R(z + \Delta(z), e^\bar{c}, z) \leq u^h(R(z_0^h)e^{-\bar{c}}, z_0^h) \quad (104)
\]

The final, strict, inequality uses the fact that \( R(z_0^h) > 0 \), as established above.

Since we have shown that \( h \) strictly prefers \( z_0^h \) to any \( z \not\in e^{B_{\bar{c}(\bar{c})}(z(R))} \) when facing the schedule \( R + \Delta \), \( h \)’s problem \( [\bar{c}(\bar{c})] \) can for any \( \Delta \in B_{\bar{c}(\bar{c})}(0) \) be rewritten as

\[
u^h_1((R + \Delta)(z), z) \quad \max_{z \in \mathbb{R}_{\geq 0}} \nu^h((R + \Delta)(z), z) \quad (105)
\]

Since the objective is continuous and the domain is compact, \( (105) \) has a solution, and in particular (by the comparison-to-\( z_0^h \) argument above) one within \( e^{B_{\bar{c}(\bar{c})}(z(R))} \).

We now argue that, for any \( \Delta \in B_{\bar{c}}(0) \), this problem has a unique solution, where \( \bar{c} \equiv \min(\bar{c}(\bar{c}), b(\eta, R)) \) and \( b(\eta, R) \equiv \min_{\eta} \frac{1}{2 \max_{\eta}(1, B^R R)} \). To see this, consider the compensating variation function \( u^h(\cdot) \) of Assumption 3. Since—by Assumption 2—\( u^h \) is three-times continuously differentiable and \( u^h > 0 \), and—by Assumption 1—\( R \) is three-
times continuously differentiable, the implicit function theorem implies that \( v \) is three-times continuously differentiable within \( B_{b,h}(\log z_0^h) \). Note that, for any \( \Delta \in \Delta \) with \( ||\Delta||<1 \) and \( z \in e^{B_{b,h}(\log z_0^h)} \), (27) implies that
\[
 u^h\left( (R(z) + \Delta(z)) e^{v^h(\log z)} - \tilde{\Delta}(\log z), z \right) = u^h\left( R(z_0^h), z_0^h \right),
\]
where \( \tilde{\Delta}(\tilde{z}) \equiv \log \left( 1 + \frac{\Delta(e^{\tilde{z}})}{R(e^{\tilde{z}})} \right) \).

Note that \( \tilde{\Delta}(\log z) \) is three-times continuously differentiable in \( B_{b,h}(\log z_0^h) \) since \( R(z) > 0 \) and \( \Delta \) and \( R \) are three-times continuously differentiable on \( \mathbb{R}_{>0} \) by Assumption 1 and the definition of \( \Delta \).

Combining (106) with (105) implies that, for \( \Delta \in B_{\bar{\Delta}(\log z_0^h)} \), it therefore suffices to show that the objective \( v^h(\cdot) - \tilde{\Delta}(\cdot) \) is strictly convex on \( B_{b,h}(\log z_0^h) \). Since we know by assumption that \( v^h(\cdot) \geq \eta \), it suffices to show that \( |\tilde{\Delta}'(\cdot)| < \eta \). To this end, we compute the derivatives of \( \tilde{\Delta} \) below.

\[
\tilde{\Delta}'(\tilde{z}) = \frac{\Delta''(e^{\tilde{z}}) e^{\tilde{z}}}{R(e^{\tilde{z}})} - \frac{R''(e^{\tilde{z}}) e^{\tilde{z}}}{R(e^{\tilde{z}})} \frac{\Delta(e^{\tilde{z}})}{R(e^{\tilde{z}})}
\]
\[
\tilde{\Delta}''(\tilde{z}) = \left( \frac{\Delta''(e^{\tilde{z}}) e^{\tilde{z}}}{R(e^{\tilde{z}})} - \frac{R''(e^{\tilde{z}}) e^{\tilde{z}}}{R(e^{\tilde{z}})} \frac{\Delta(e^{\tilde{z}})}{R(e^{\tilde{z}})} \right) \left( \frac{\Delta'(e^{\tilde{z}}) e^{\tilde{z}}}{R(e^{\tilde{z}})} - \frac{R'(e^{\tilde{z}}) e^{\tilde{z}}}{R(e^{\tilde{z}})} \frac{\Delta(e^{\tilde{z}})}{R(e^{\tilde{z}})} \right)
\]
\[
\tilde{\Delta}'''(\tilde{z}) = \left( \frac{\Delta'''(e^{\tilde{z}}) e^{\tilde{z}}}{R(e^{\tilde{z}})} - \frac{R'''(e^{\tilde{z}}) e^{\tilde{z}}}{R(e^{\tilde{z}})} \frac{\Delta'(e^{\tilde{z}}) e^{\tilde{z}}}{R(e^{\tilde{z}})} \right) \left( \frac{\Delta'(e^{\tilde{z}}) e^{\tilde{z}}}{R(e^{\tilde{z}})} - \frac{R'(e^{\tilde{z}}) e^{\tilde{z}}}{R(e^{\tilde{z}})} \frac{\Delta'(e^{\tilde{z}}) e^{\tilde{z}}}{R(e^{\tilde{z}})} \right)
\]

It remains to show that these derivatives are bounded for small enough tax deviations \( \Delta \). To this end, recall first that by Assumption 1, there exists \( B^R \geq 0 \) such that for all \( z \in \mathbb{R}_{>0} \), \( |R'(z)| \leq B^R |R(z)| \) and \( |R''(z)| \leq B^R |R'(z)| \). Moreover, recall that for any constant \( b > 0 \), if \( ||\Delta|| < b \), then \( |\Delta(z)| \leq b |R(z)| \), with strict inequality if \( |R(z)| > 0 \), and \( |\Delta'(z)| \leq b |R'(z)| \) and \( |\Delta''(z)| \leq b |z R''(z)| \), with strict inequality if \( |R'(z)| > 0 \). Combining these observations, we may note that for any \( \Delta \in B_b(0) \), \( z \in e^{B_{b,h}(\log z_0^h)} \),
\[
\frac{|\Delta(z)|}{R(z)} < b, \quad \frac{|\Delta'(z) z^2}{R(z)} \leq b B^R, \quad \frac{|\Delta''(z) z^3}{R(z)} \leq b B^R, \quad \frac{|\Delta'''(z) z R'(z) z}{R(z)} \leq b(B^R)^2,
\]
\[
\frac{|R''(e^{\tilde{z}}) e^{2\tilde{z}} \Delta(e^{\tilde{z}})|}{R(e^{\tilde{z}})} \leq b(B^R)^2, \quad \frac{|R'(e^{\tilde{z}}) e^{\tilde{z}} \Delta'(e^{\tilde{z}}) e^{\tilde{z}}|}{R(e^{\tilde{z}})} \leq b B^R,
\]
\[
\frac{|R'(e^{\tilde{z}}) e^{\tilde{z}} \Delta'(e^{\tilde{z}}) e^{\tilde{z}}|}{R(e^{\tilde{z}})} \leq b B^R, \quad \frac{|R'(e^{\tilde{z}}) e^{\tilde{z}} \Delta'(e^{\tilde{z}}) e^{\tilde{z}}|}{R(e^{\tilde{z}})} \leq b(B^R)^2.
\]

Combining these inequalities implies that \( \tilde{\Delta}''(\tilde{z}) \) is uniformly bounded by \( \eta \) across all
Toward an application of the implicit function theorem, define $F$ space. Next we claim continuously differentiable in $B$ on $\Delta$. First note that $\Delta$ is defined as in (106). This implies that, for $\Delta \in B_{\min(\hat{z}(\epsilon), b(\eta, R))}(0)$, $h$’s problem at $R + \Delta$ has a unique solution. (Note that $u^h(R)$ is therefore $> -\infty$, a fact we will use throughout.)

Differentiability of labor supply function

So far, we have shown the existence of a unique labor supply function $z^h(R + \Delta) : R + B_{\min(\hat{z}(\epsilon), b(\eta, R))}(0) \rightarrow e^{B_{h, \eta}(\log z_0^h)}$ in a neighborhood of an initial tax schedule $R$. We will now show that $z^h(R + \Delta)$ is twice continuously differentiable.

To start, recall from (107) that

$$z^h(R + \Delta) = \arg\min_{z \in e^{B_{h, \eta}(\log z_0^h)}} v^h(\log z) - \tilde{\Delta}(\log z).$$

(112)

Recall we have already argued that $v^h(\cdot) - \tilde{\Delta}(\cdot)$ is three times continuously differentiable on $B_{h, \eta}(\log z_0^h)$.

As we have shown that $z^h(R + \Delta)$ exists and is in $e^{B_{h, \eta}(\log z_0^h)}$, we have the first order condition:

$$\frac{d}{d \log z} \left( v^h - \tilde{\Delta} \right) (\log z^h(R + \Delta)) = 0.$$ 

(113)

Toward an application of the implicit function theorem, define $F : (B_{\min(\hat{z}(\epsilon), b(\eta, R))}(0) \subset \Delta) \times (B_{h, \eta}(\log z_0^h) \subset \mathbb{R}) \rightarrow \mathbb{R}$ by

$$F(\Delta, \log z) = \frac{d}{d \log z} \left( v^h(\log z) - \log \left( 1 + \frac{\Delta(z)}{R(z)} \right) \right).$$ 

(114)

First note that $F$ is a map from a product of open subsets of Banach spaces to a Banach space. Next we claim $F$ is twice-continuously Frechet differentiable. That $F$ is twice-continuously differentiable in $\log z$ follows from that $v^h$, $\Delta$, and $R$ are all three-times continuously differentiable, as discussed above. It therefore suffices to show that $F$’s derivatives in $\Delta$ (including the cross-partial with $\log z$) exist and are—at each $\Delta$, $\log z$—bounded (over all directions in which the derivatives in $\Delta$ may be taken). To see this,

\[ \|\Delta\| < b(\eta, R). \] To see this, note that for any $b \in (0, 1)$

$$\tilde{\Delta}''(\tilde{z}) < \frac{5b(B^R)^2 + 3bB^R}{1 - b} + \frac{2bB^R}{(1 - b)^2} \leq 10 \frac{b}{(1 - b)^2} \max(1, B^R)^2$$

(110)

In particular, $\tilde{\Delta}''(z) < \eta$ if $b < \min \left( \frac{1}{2}, \frac{\eta}{10 \max(1, B^R)^2} \right) \equiv b(\eta, R)$\textsuperscript{85}

Taking stock, if $\Delta \in B_{b(\eta, R)}(0)$ then $v^h(\tilde{z}) + \tilde{\Delta}(\tilde{z})$ is strictly convex on $B_{h, \eta}(\log z_0^h)$, where $\tilde{\Delta}(\cdot)$ is defined as in (106). This implies that, for $\Delta \in B_{\min(\hat{z}(\epsilon), b(\eta, R))}(0)$, $h$’s problem at $R + \Delta$ has a unique solution. (Note that $u^h(R)$ is therefore $> -\infty$, a fact we will use throughout.)

\textsuperscript{85}Doing out the algebra, let $\bar{b} = \min \left( \frac{1}{2}, \frac{\eta}{10 \max(1, B^R)^2} \right)$. Then for any $b < \bar{b}$,

$$\frac{b}{(1 - b)^2} \max(1, B^R)^2 < 10 \frac{\bar{b}}{(1 - \bar{b})^2} \max(1, B^R)^2 \leq 40 \bar{b} \max(1, B^R)^2 \leq \eta$$

(111)
we compute

\[ F(\Delta, \log z) = v^h(\log z) - \frac{\Delta'(z) \log R(z)}{1 + \Delta(z) R(z)} d \log R(z) \]

\[ D_\Delta F(\Delta, \log z) = \left( - \frac{\tilde{\Delta}'(z)}{1 + \Delta(z) R(z)} + \frac{\Delta'(z) \log R(z)}{1 + \Delta(z) R(z)} \right) d \log R(z) \]

\[ D^2_{\Delta \Delta} F(\Delta, \log z) = \left( \frac{\tilde{\Delta}'(z) \log R(z)}{1 + \Delta(z) R(z)} - \frac{\Delta'(z) \log R(z)}{1 + \Delta(z) R(z)} \right) d \log R(z) \]

\[ D^2_{\Delta \log z} F(\Delta, \log z) = \left( \frac{\tilde{\Delta}'(z) \log R(z)}{1 + \Delta(z) R(z)} - \frac{\Delta'(z) \log R(z)}{1 + \Delta(z) R(z)} \right) d \log R(z) \]

It is immediate from Assumption [1] the definition of \( \Delta \), and the fact that \( \Delta \in B_{2,2}(\mathbf{0}) \) that all of the derivatives above are bounded proportionally to \( ||\tilde{\Delta}|| \) and \( ||\tilde{\Delta}|| \) as appropriate. \( F \) is therefore Frechet differentiable.

Finally, recall we have shown that \( D_{\log z} F(\Delta, \log z) = \frac{d}{d \log z}\left[ v^h(\log z) - \log \left( 1 + \frac{\Delta(z)}{R(z)} \right) \right] > 0 \). The implicit function theorem for Banach spaces therefore implies that a continuously differentiable solution \( \log z^h(\Delta) \in B_{2,2}(\log z^h(0)) \) to the equation \( F(\Delta, \log z(\Delta)) \) exists on \( \Delta \in B_{2,2}(\mathbf{0}) \) and satisfies \( D_{\log z} \log z(\Delta) = - (D_{\log z} F(\Delta, \log z(\Delta)))^{-1} D_{\tilde{\Delta}} F(\Delta, \log z(\Delta)) \).

This derivative is bounded over all \( \tilde{\Delta} \) proportionally to \( ||\tilde{\Delta}|| \) by the arguments above, so it is a Frechet derivative. Moreover, note that—since \( F \) is twice-Frechet differentiable, \( \log z^h(\Delta) \) has a second derivative given by

\[ D^2_{\Delta \Delta} \log z(\Delta) = (D_{\log z} F(\Delta, \log z(\Delta)))^{-1} D_{\log z} F(\Delta, \log z(\Delta)) D_{\Delta} F(\Delta, \log z(\Delta)) \]

\[ \quad - (D_{\log z} F(\Delta, \log z(\Delta)))^{-1} D_{\tilde{\Delta}} F(\Delta, \log z(\Delta)) \]

\[ (115) \]

\[ \text{See, e.g., Martinsson} \]
Since by the arguments above this is bounded over all $\Delta, \hat{\Delta}$ proportionally to $||\Delta|||\hat{\Delta}||$, it is a Frechet derivative as well.

Recalling from the previous step of the proof that the equation $F(\Delta, \log z) = 0$ has a unique solution on its domain, we identify $e^{\log z(\Delta)}$ with $z^h(R + \Delta)$ and conclude that $z^h(\cdot)$ is twice continuously (Frechet) differentiable in $R + B_{\min(\hat{\Delta}, (q, R))}(0)$.

**Measurability of labor supply**

Fixing any $\hat{R} \in R + B_{\min(\hat{\Delta}, (q, R))}(0)$, we wish to show that $z^h(\hat{R})$—which recall we have just shown is well-defined—is measurable in $h$. This argument, written out below, is a straightforward implication of the measurable maximum theorem as stated in Aliprantis and Border [2006]. The theorem states that, if $\mu$ is a weakly measurable function (continuous in its first argument and measurable in its second), then the arg max function $\mu(h) \equiv \arg \max_{z \in \Gamma(h)} f(z, h)$ admits a measurable selector.

First define $\Gamma : \mathcal{H} \mapsto \mathbb{R}_{>0}$ by $h \mapsto e^{\int_{\hat{A}} (\log z^h)}$. Note this correspondence has non-empty and compact values. To see it is moreover weakly measurable, note that it may be obtained by the composition of the (by assumption and our use of the canonical product measure) measurable function $h \mapsto (\log z^h, \hat{e}^h)$ with the correspondence $\hat{\Gamma} : (\hat{z}, \epsilon) \mapsto e^{\int_{\hat{\Delta}} (\hat{z})}$. It therefore suffices to show that this correspondence $\hat{\Gamma}$ is weakly measurable. To see this fix any (WLOG non-empty) open set $A \subset \mathbb{R}_{>0}$ and let $\hat{A}$ be a dense, countable subset of $A$.

Then

$$
\hat{\Gamma}^{-1}(A) = \left\{\left(\hat{z}, \epsilon\right) \in \mathbb{R} \times \mathbb{R}_{0} \mid \exists \hat{a} \in \hat{A}, \hat{a} \in e^{\int_{\hat{\Delta}} (\hat{z})}\right\}
= \left\{\left(\hat{z}, \epsilon\right) \in \mathbb{R} \times \mathbb{R}_{0} \mid \exists \hat{a} \in \hat{A}, \hat{a} \in e^{\int_{\hat{\Delta}} (\hat{z})}\right\}
= \bigcup_{\hat{a} \in \hat{A}} \Gamma^{-1}(\{\hat{a}\})
$$

where in the second line we have used $\hat{A}$’s openness and $\hat{A}$’s density in $\hat{A}$. Since $\hat{A}$ is countable it suffices to show each $\Gamma^{-1}(\{\hat{a}\})$ is measurable.

Indeed, $\Gamma^{-1}(\{\hat{a}\}) = \emptyset$ (which

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87 Precisely, the Theorem also requires that $\mathbb{R}_{>0}$ is a separable metrizable space (it is), and $\mathcal{H}$ and its associated event space constitute a measurable space (they do).

88 Any non-empty subset $A$ of $\mathbb{R}$ admits a countable, dense subset. Here we provide a proof: Fix any $a_0 \in A$. For all $n \in \mathbb{Z}$, $m, k \in \mathbb{N}$, $k \leq m$; define $x_{m,k}$ as follows: If $[n + \frac{k - 1}{m}, n + \frac{k}{m}]$ intersects $A$, let $x_{m,k} = a_0$. We claim that $\hat{A} = \bigcup_{n \in \mathbb{Z}, k, m \in \mathbb{N}, k \leq m} \{x_{m,k}\}$ is a countable (obviously) subset (obviously) of $A$ which is dense in $A$. To see $\hat{A}$ is dense in $A$, note that for all $a \in A$, $a$ is contained in some interval $[n, n+1]$ for $n \in \mathbb{Z}$, and moreover for all $m \in \mathbb{N}$, there exists $k \in \mathbb{N}, k \leq m$ for which $a \in [n + \frac{k - 1}{m}, n + \frac{k}{m}]$. Therefore, for all $m \in \mathbb{N}$, there exists a point $x_{m,k} \in \hat{A}$ within $\frac{1}{m}$ of $a$. So $a$ is a limit point of $\hat{A}$.

89 This argument is similar to the one presented in Himmelberg [1975], available at [http://repository.iias.ac.in/90958/1/90958.pdf](http://repository.iias.ac.in/90958/1/90958.pdf)
is measurable) if \( \tilde{a} \leq 0 \) and if \( \tilde{a} > 0 \),

\[
\tilde{\Gamma}^{-1}(\{\tilde{a}\}) = \{(\tilde{z}, \epsilon) \in \mathbb{R} \times \mathbb{R}_{>0} \mid \tilde{z} - \epsilon \leq \log a \leq \tilde{z} + \epsilon\}
\]

\[
= \bigcup_{n=0,1,2,...,m=1,2,3,...} \left[ \log \tilde{a} - n - \frac{1}{m}, \log \tilde{a} + n + \frac{1}{m} \right] \times \left[ n + \frac{1}{m}, 1 + n + \frac{1}{m} \right]
\]

\[
\cup \left[ \log \tilde{a} - n - \frac{1}{m}, \log \tilde{a} + n + \frac{1}{m} \right] \times \left[ -1 - n - \frac{1}{m}, -n - \frac{1}{m} \right]
\]

which is Lebesgue-product-measurable since it is a countable union of rectangles.

Second, define \( f : \mathbb{R}_{>0} \times \mathcal{H} \to \mathbb{R} \) by \((z, h) \mapsto u^h(\tilde{R}(z), z)\). \( f \) is well-defined by Assumption 2 since \( ||\tilde{R} - R|| < 1 \) and \( z > 0 \) implies \( \tilde{R}(z) > 0 \). Moreover \( f \) is continuous since \( u^h \) and \( \tilde{R} \) are continuous, and \( h \)-measurable since \( u^h \) is.

Theorem 18.19 of Aliprantis and Border [2006] implies that the maximization problem

\[
\max_{z \in \Gamma(h)} f(z, h)
\]

has an argmax correspondence that admits a measurable selector \( s(h) \in \mathbb{R}_{>0} \). Since we have already established that this maximization problem has a unique solution for each \( h \), it must be that \( s(h) = z^h(\tilde{R}) \). We conclude that \( z^h(\tilde{R}) \) is measurable in \( h \).

**First derivative of labor supply**

We have already established each household’s labor supply function \( z^h(\tilde{R}) \) is well-defined and has two continuous Frechet derivatives when \( \tilde{R} \in R + B_\delta(0) \). While it is not strictly necessary to compute these derivatives in order to complete the proof of the Lemma, this section computes explicit expressions for them, which will be used in a later step of the proof. We therefore now fix such a \( \tilde{R} \) and a household \( h \in \mathcal{H} \), and will compute these Frechet derivatives.

We compute the first derivative by totally differentiating \( h \)’s MRS condition, since by continuous differentiability of preferences and taxes and since \( z^h(\tilde{R}) > 0 \), this always holds at the solution to \( h \)’s problem. Fixing any direction \( \Delta \in \Delta \) and recalling \( M^h(c, z) \equiv -u^h_c(c, z) / u^h_z(c, z) \), we have

\[
M^h \left( \tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}) \right) = \tilde{R}'(z^h(\tilde{R}))
\]

\[
\frac{d \log M^h(c, z)}{d \log c} \bigg|_{c = \tilde{R}(z^h(\tilde{R}))} \left( \frac{\Delta(z^h(\tilde{R}))}{R'(z^h(\tilde{R}))} + \frac{d \log \tilde{R}(z)}{d \log z} \right)_{z = z^h(\tilde{R})} D_{\Delta} \log z^h(\tilde{R})
\]

\[
+ \frac{d \log M^h(c, z)}{d \log z} \bigg|_{c = \tilde{R}(z^h(\tilde{R}))} \left( \frac{\Delta'(z^h(\tilde{R}))}{R'(z^h(\tilde{R}))} + \frac{d \log \tilde{R}(z)}{d \log z} \right)_{z = z^h(\tilde{R})} D_{\Delta} \log z^h(\tilde{R})
\]

\[
D_{\Delta} \log z^h(\tilde{R}) = \frac{d \log M^h(c, z^h(\tilde{R}))}{d \log c} \frac{\Delta(z^h(\tilde{R}))}{R'(z^h(\tilde{R}))} + \frac{\Delta'(z^h(\tilde{R}))}{R'(z^h(\tilde{R}))} - \frac{d \log \tilde{R}(z^h(\tilde{R}))}{d \log z} \bigg|_{z = z^h(\tilde{R})},
\]

where we may divide through by the denominator in the last line since—were it zero—then \( D_{\Delta} \log z^h(\tilde{R}) \) would not exist for some \( \Delta \), and we have already established that \( D_{\Delta} \log z^h(\tilde{R}) \) exists. Note that the log of \( M^h \), \( \tilde{R}(z^h) \), and \( \tilde{R}'(z^h) \) are well defined since \( z^h(\tilde{R}) > 0 \) implies \( \tilde{R}(z^h), \tilde{R}'(z^h) > 0 \) and since by Assumption 2 this implies \( u^h_c(e^h, z^h) > 0 \),
This expression in mind, we define

\[
\eta^h(\tilde{R}) = \frac{d \log M^h(c^h, z^h)}{d \log c^h} \cdot \frac{d \log M^h(c^h, z^h)}{d \log z^h} + \frac{d \log M^h(c^h, z^h)}{d \log c^h} \cdot \frac{d \log \tilde{R}(z^h)}{d \log z^h} \quad z^h = \tilde{z}(\tilde{R}), \quad c^h = \tilde{c}(\tilde{R})^h
\]

(121)

\[
\varepsilon^h(\tilde{R}) = \frac{d}{d \log \tilde{R}} \left[ \frac{\Delta(z^h(\tilde{R}))}{\tilde{R}(z^h(\tilde{R}))} \right] + \frac{\Delta'(z^h(\tilde{R}))}{\tilde{R}'(z^h(\tilde{R}))} \quad z^h = \tilde{z}(\tilde{R}), \quad c^h = \tilde{c}(\tilde{R})^h
\]

(122)

Note that—by our expression for \( D_\Delta \log z^h(\tilde{R}) \)—this coincides with the definition in the main text, in (29), but formally treats the set of feasible deviations. In terms of these elasticities the first Frechet derivative of labor supply can be written as

\[
D_\Delta \log z^h(\tilde{R}) = \eta^h(\tilde{R}) \frac{\Delta(z^h(\tilde{R}))}{\tilde{R}(z^h(\tilde{R}))} + \varepsilon^h(\tilde{R}) \frac{\Delta'(z^h(\tilde{R}))}{\tilde{R}'(z^h(\tilde{R}))}
\]

Aside: positivity of compensated elasticity

Since they are the second derivatives of the expenditure function, compensated elasticities are always weakly positive. For the reader who is not convinced that \( \varepsilon^h(\tilde{R}) \) as defined above is a compensated elasticity in the traditional sense, we suggest [Scheuer and Werning 2018]. However, for completeness we also provide below an explicit proof that \( \varepsilon^h(\tilde{R}) \geq 0 \).

Fix \( h \in H \). Let \( z^h = z^h(\tilde{R}) \). Since \( z^h(\tilde{R}) \) solves \( h \)'s problem at \( \tilde{R} \),

\[
0 = \left. \frac{d}{dz} \right|_{z=z^h} u^h(\tilde{R}(z), z)
= u^h_c(\tilde{R}(z^h), z^h) \left[ \tilde{R}'(z^h) - M^h(\tilde{R}(z^h), z^h) \right]
\]

(123)

where here we have used that \( \tilde{R}'(z^h) = M^h(\tilde{R}(z^h), z^h) > 0 \).

We conclude that

\[
\frac{d \log \tilde{R}'(z^h)}{d \log z^h} - \frac{d \log M^h(c^h, z^h)}{d \log c^h} \frac{d \log \tilde{R}(z^h)}{d \log z^h} - \frac{d \log M^h(c^h, z^h)}{d \log z^h} \geq 0,
\]

implying \( \varepsilon^h(\tilde{R}) \geq 0 \).
To compute the second derivative of labor supply, we differentiate the first derivative along an arbitrary direction \( \Delta \in \mathbb{R} \).

\[
D_{\Delta}^2 \log z^h(\tilde{R}) = D_{\Delta} \left( \frac{\eta^h(\tilde{R}) \Delta z^h(\tilde{R})}{R(z^h(\tilde{R}))} + \epsilon^h(\tilde{R}) \frac{\Delta' z^h(\tilde{R})}{R(z^h(\tilde{R}))} \right)
\]

\[
= D_{\Delta} \left( \frac{\Delta \eta^h(\tilde{R})}{R^h} + \eta^h \left[ \frac{\Delta^2 \eta^h}{R^h} - \frac{\Delta \eta^h \log \tilde{R}^h}{R^h} \frac{d \log \tilde{R}^h}{d \log \tilde{R}^h} \right] D_{\Delta} \log z^h(\tilde{R}) \right)
\]

\[
+ D_{\Delta} \left( \frac{\Delta \epsilon^h(\tilde{R})}{R^h} + \epsilon^h \left[ \frac{\Delta^2 \epsilon^h}{R^h} - \frac{\Delta \epsilon^h \log \tilde{R}^h}{R^h} \frac{d \log \tilde{R}^h}{d \log \tilde{R}^h} \right] D_{\Delta} \log z^h(\tilde{R}) \right)
\]

(124)

where above, and in the equations below, variables are evaluated at \( \tilde{R}, z^h = z^h(\tilde{R}) \), and/or \( \epsilon^h = \tilde{R}(z^h) \) as relevant; and \( \tilde{R}^h \) is shorthand for \( \tilde{R}(z^h) \) and similar.

Next, we compute \( D_{\Delta} \epsilon^h(\tilde{R}) \) and \( D_{\Delta} z^h(\tilde{R}) \) in isolation by differentiating (121):

\[
D_{\Delta} \epsilon^h(\tilde{R}) = - (\epsilon^h)^2 \left[ D_{\Delta} \log z^h + \left( \frac{\eta^h}{R^h} + \epsilon^h \frac{\eta^h}{R^h} \right) D_{\Delta} \log z^h \right]
\]

\[
+ (\epsilon^h)^2 \left[ \frac{\eta^h}{R^h} + \epsilon^h \frac{\eta^h}{R^h} \right] D_{\Delta} \log z^h
\]

(125)

Note that the terms \( \eta^h \epsilon^h \frac{d \log R^h}{d \log z} \) \( D_{\Delta} \log z^h \), \( (\epsilon^h)^2 \frac{d \log R^h}{d \log z} \) \( D_{\Delta} \log z^h \), \( \epsilon^h \frac{\eta^h}{R^h} + \epsilon^h \frac{\eta^h}{R^h} \)

and \( \epsilon^h \frac{\eta^h}{R^h} \frac{\eta^h}{R^h} \) correspond to changes in \( \epsilon^h(\tilde{R}) \) through changes in labor supply, whereas the complementary terms correspond to changes in \( \epsilon^h(\tilde{R}) \) directly through changes in the tax schedule. By the earlier expression for \( D_{\Delta} \log z^h \), we can decompose the terms operating through changes in labor supply into income and compensated effects. For later use, we
Define $\varepsilon^{+h}$ to be the compensated component, i.e.

$$
\varepsilon^{+h}(\tilde{R}) \equiv \varepsilon^{h+1}(\tilde{R}) + \eta^h(\varepsilon^h)^2 \frac{d^2 \log \tilde{R}(z)}{d \log z^2} + (\varepsilon^h)^3 \frac{d^2 \log \tilde{R}'(z)}{d \log z^2}
$$

(126)

$$
D_\Delta \eta^h(\tilde{R}) = - \frac{d \log M^h}{d \log c} \frac{d}{d \log z} \left[ \frac{d}{d \log z} \left( \frac{d \log M^h}{d \log z} \right) \right] - \frac{d \log \tilde{R}^h}{d \log z} - \frac{d \log \tilde{R}^h}{d \log z} \frac{d}{d \log z} \left[ \frac{d}{d \log z} \left( \frac{d \log M^h}{d \log z} \right) \right]
$$

(127)

where $\eta_{+1}^h = - \eta^h(\varepsilon^h)^2 \left[ \frac{d}{d \log z} \left( \frac{d \log M^h}{d \log z} \right) \right] + \frac{d}{d \log z} \left( \frac{d \log \tilde{R}^h}{d \log z} \right) + \frac{d}{d \log z} \left( \frac{d \log M^h}{d \log z} \right) \eta^h

Note that $\varepsilon_{+1}^h = \eta_{+1}^h$. This reflects that they are both (essentially) second derivatives, but with opposite orders of differentiation.
Finally, we substitute these expressions back into (124). Simplifying, we obtain:

\[
D_\Delta^2 \log x^h(\tilde{R}) = \eta^h \left[ \left( \frac{\Delta h \log \tilde{R}}{R^h} - \frac{\Delta h \log \tilde{R}}{R^h} \right) D_\Delta \log x^h - \frac{\Delta h \log \tilde{R}}{R^h} + \varepsilon^h \left[ \left( \frac{\Delta h \log \tilde{R}}{R^h} - \frac{\Delta h \log \tilde{R}}{R^h} \right) D_\Delta \log x^h - \frac{\Delta h \log \tilde{R}}{R^h} \right] \right] + \eta^h \left[ \frac{\Delta h \log \tilde{R}}{R^h} - \frac{\Delta h \log \tilde{R}}{R^h} \right] D_\Delta \log x^h + \eta^h \left[ \frac{\Delta h \log \tilde{R}}{R^h} - \frac{\Delta h \log \tilde{R}}{R^h} \right] + \eta^h \left[ \frac{\Delta h \log \tilde{R}}{R^h} - \frac{\Delta h \log \tilde{R}}{R^h} \right]
\]

(128)

Measurability of labor supply derivatives

Fix any $\tilde{R} \in R + B(0)$ and $\Delta, \tilde{\Delta} \in \Delta$. We wish to show $D_\Delta z^h(\tilde{R})$ and $D_{\Delta \tilde{\Delta}} z^h(\tilde{R})$ are measurable in $h$. Because

\[
D_\Delta z^h(\tilde{R}) = z^h(\tilde{R}) D_\Delta \log z^h(\tilde{R})
\]

(131)

and we have shown $z^h(\tilde{R})$ is measurable, it suffices to show that $D_\Delta \log z^h(\tilde{R})$ and $D_{\Delta \tilde{\Delta}} \log z^h(\tilde{R})$ are measurable in $h$.

We begin with $D_\Delta \log z^h(\tilde{R})$. Recall our expression, (122), for this term. Since $z^h(\tilde{R})$ is measurable, $\Delta$ and $\tilde{R}$ are continuous in $z$, and $\tilde{R}(z^h(\tilde{R})) > 0$ and $\tilde{R}(z^h(\tilde{R})) > 0$ for all
Since with respect to the product measure, the set of triples $(h, c, z)$ from that \(\tilde{\delta} < \in H\) is measurable Frechet derivatives in \(H\). Properties of post-tax income
d\(h\) except for \(d\) are continuous functions of functions which we have already shown to be measurable, we refer to their definitions in (125) and (127). All of the elasticities \(\Delta\) and \(\Delta'\) are measurable. Their MRS curvature terms are measurable because they are continuous functions of the first and second derivatives of preferences, both of which are continuous and measurable in \((h, c, z)\) — and because for any \((c, z)\)-continuous and \((h, c, z)\)-measurable function \(f(h, c, z)\), \(f(h, R(z^h(\tilde{R})), z^h(\tilde{R}))\) is measurable in \(h\).

Since—as we have also already argued—the denominators of both expressions in (121) are non-zero for all \(h \in H\), both elasticities are continuous functions of measurable functions and so are measurable.

We now argue that \(D_{\tilde{\Delta}}^2 \log z^h(\tilde{R})\) is measurable in \(h\). Recall our expression, (130), for this term. On top of the terms we have already considered in showing the first derivative is measurable, we now must also show that \(\frac{\Delta''(z^h(\tilde{R}))z^h(\tilde{R})}{R''(z^h(\tilde{R}))}\), and each of the super-elasticities are measurable. \(\frac{\Delta''(z^h(\tilde{R}))z^h(\tilde{R})}{R''(z^h(\tilde{R}))}\) (and similarly for \(\tilde{\Delta}\)) is measurable because \(\Delta''\) and \(\tilde{R}'\) are continuous and \(z^h(\tilde{R})\) are measurable. To show the super-elasticities are measurable, we refer to their definitions in (125) and (127). All of the elasticities are continuous functions of functions which we have already shown to be measurable, except for \(\frac{d}{d \log c} \left( \frac{d \log M^h(\tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}))}{d \log z} \right)\) and \(\frac{d}{d \log c} \left( \frac{d \log M^h(\tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}))}{d \log z} \right)\). Note that these are both continuous functions of the first, second, and third derivatives of preferences, all of which are continuous and measurable in \((h, c, z)\). By our earlier observations, this implies these terms are measurable as well.

Properties of post-tax income

It remains to show that, for \(h \in H\), \(\tilde{R} \in R + B_3(0)\), post-tax income \(\tilde{R}(z^h(\tilde{R}))\) is well-defined, measurable, and strictly positive, and that it has two continuous and \(H\)-measurable Frechet derivatives in \(\tilde{R}\).

These properties are all inherited from those of \(z^h(\tilde{R})\). Well-definedness is immediate. Measurability follows from that \(z^h(\tilde{R})\) is measurable and \(\tilde{R}\) is continuous. Positivity follows from that \(z^h(\tilde{R}) > 0\), our earlier observation that \(R(z) > 0\) for all \(z > 0\), and that \(\delta < \frac{1}{2}\). Finally, the existence, continuity, and measurability of Frechet derivatives follows from that \(\tilde{R}\) is twice-contuously differentiable and that \(z^h(\tilde{R})\) has two continuous and \(H\)-measurable Frechet derivatives.

\(^{90}\)For any measurable set \(A\) in the image of \(f\), \(f^{-1}(A)\) is a measurable set consisting of \((h, c, z)\) triples. Since \(\tilde{R}(z^h(\tilde{R}))\) and \(z^h\tilde{R}\) are measurable, and since component-wise measurable functions are measurable with respect to the product measure, the set of triples \((h_1, h_2, h_3)\) such that \((h_1, \tilde{R}(z^{h_2}(\tilde{R})), z^{h_3}(\tilde{R})) \in f^{-1}(A)\) is also measurable. It remains to show that the diagonal of the set of these triples \((h_1, h_2, h_3)\) is a measurable set in \(H\). For this it suffices to show that the diagonal function \(h \mapsto (h, h, h)\) is measurable. To see this note that it suffices to check on a generic generating element of the product measure, e.g. \(H \times H' \times H''\), whose inverse image is simply \(H \cap H \cap H' \cap H''\); this is measurable if \(H\), \(H'\), and \(H''\) each are.

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E.3 Proof of Lemma 3

We complete the proof in several steps: First we provide bounds on the first and second derivatives of log labor supply, second we show that labor supply and its derivatives are integrable, and third we show that post-tax income and its derivatives are bounded and integrable. Throughout we fix $\delta > 0$ to be smaller than the values referred to in both Assumption 4 and Lemma 2.

First and second derivatives of labor supply: bounding

We now show that there exist uniform upper bounds on the Frechet derivatives $D\log z^h(\tilde{R})$ and $D^2\log z^h(\tilde{R})$ (as linear maps) across all $\tilde{R} \in R + B_\delta(0)$, $h \in H$. WLOG we will fix $\Delta, \tilde{\Delta} \in \Delta$ with $\|\Delta\| = \|\tilde{\Delta}\| = 1$ and the show existence of uniform bounds on $D_\Delta \log z^h(\tilde{R})$ and $D^2_\Delta \log z^h(\tilde{R})$.

By the expressions (122) and (130) for the first and second derivatives of labor supply, it suffices to provide uniform bounds on

- all elasticities and super-elasticities;
- across all $z > 0$, the log-tax-change terms $\frac{\Delta(z)}{\tilde{R}(z)} \frac{\Delta'(z)}{\tilde{R}'(z)}$ and $\frac{\Delta''(z)}{\tilde{R}''(z)}$ (those for $\tilde{\Delta}$ are analogous); and

- across all $z > 0$, the tax curvature terms $\frac{d}{d\log z} \frac{d \log \tilde{R}(z)}{d \log z}$, $\frac{d}{d\log z} \frac{d \log \tilde{R}'(z)}{d \log z}$, $\frac{d}{d\log z} \frac{d \log \tilde{R}''(z)}{d \log z}$, and $\frac{1}{\tilde{R}''(z)}$.

The first bullet is ensured by Assumption 4.

Next, consider the second bullet. Since (from the Lemma we are trying to prove) we need only show this is true for sufficiently small $\delta$, we may WLOG assume $\delta < \frac{1}{2}$. In this case, $\tilde{R} \in B_\delta(0)$ and $\|\Delta\| = 1$ imply

$$\left| \frac{\Delta(z)}{\tilde{R}(z)} \right| \leq \frac{1}{2} \frac{\Delta(z)}{\tilde{R}(z)} \leq 2, \quad \left| \frac{\Delta'(z)}{\tilde{R}'(z)} \right| \leq 1, \quad \left| \frac{\Delta''(z)}{\tilde{R}''(z)} \right| \leq B R \left| \frac{\tilde{R}'(z)}{\tilde{R}''(z)} \right| \leq B R \left| \frac{\tilde{R}'(z)}{\tilde{R}''(z)} \right| = 2B R. \tag{132}$$

Finally, consider the third bullet. Again, using that $\delta < \frac{1}{2}$, we have

$$\left| \frac{d}{d\log z} \frac{d \log \tilde{R}(z)}{d \log z} \right| \leq \frac{1}{2} \frac{\tilde{R}'(z)}{\tilde{R}''(z)} \leq \frac{1}{2} \frac{\tilde{R}'(z)}{\tilde{R}''(z)} \leq \frac{1}{2} \frac{\tilde{R}'(z)}{\tilde{R}''(z)} \leq 3B R,$$

$$\left| \frac{d}{d\log z} \frac{d \log \tilde{R}'(z)}{d \log z} \right| \leq \frac{1}{2} \frac{\tilde{R}'(z)}{\tilde{R}''(z)} \leq \frac{1}{2} \frac{\tilde{R}'(z)}{\tilde{R}''(z)} \leq 3B R,$$

$$\left| \frac{d}{d\log z} \frac{d \log \tilde{R}''(z)}{d \log z} \right| \leq \frac{1}{2} \frac{\tilde{R}'(z)}{\tilde{R}''(z)} \leq \frac{1}{2} \frac{\tilde{R}'(z)}{\tilde{R}''(z)} \leq 3B R.$$
where on the penultimate line, we have used that $\frac{d^2\tilde{R}(z)}{d\log z^2} = z\tilde{R}''(z) + z^2\tilde{R}'''(z)$.

We conclude that the level and first two derivatives of log labor supply (appropriately normalized) are uniformly bounded across $h \in \mathcal{H}$ and $\tilde{R} \in R + B_\delta(0)$.

**Integrability of labor supply and its derivatives**

In this section we argue that—for any $\tilde{R} \in R + B_\delta(0)$, $\Delta, \bar{\Delta} \in \Delta — \delta^h(\tilde{R})$, $D_\Delta z^h(\tilde{R})$, and $D^2_{\Delta\Delta} z^h(\tilde{R})$ are bounded across all $h \in \mathcal{H}$ by constants times $z_0^h$ and are integrable. Since by Lemma 2, $z^h(\tilde{R})$, $D_\Delta z^h(\tilde{R})$, and $D^2_{\Delta\Delta} z^h(\tilde{R})$ are measurable, and by Assumption 4, $z^h_0$ is integrable, the bounding is sufficient for integrability. To this end, a useful observation is that:

$$D_\Delta z^h(\tilde{R}) = z^h(\tilde{R})D_\Delta \log z^h(\tilde{R})$$
$$D^2_{\Delta\Delta} z^h(\tilde{R}) = z^h(\tilde{R}) \left[ D_\Delta \log z^h(\tilde{R})D_\Delta \log z^h(\tilde{R}) + D_\Delta D^2_{\Delta\Delta} \log z^h(\tilde{R}) \right] \tag{134}$$

Since we have shown that $D_\Delta \log z^h(\tilde{R})$, $D_\Delta \log z^h(\tilde{R})$, and $D^2_{\Delta\Delta} \log z^h(\tilde{R})$ are uniformly bounded, it remains to show $z^h(\tilde{R})$ is bounded in absolute value by a constant times $z_0^h = z^h(\tilde{R})$.

Indeed, for all $\tilde{R} \in R + B_\delta(0)$, since $z^h(\tilde{R}) > 0$ and $\log z^h(\tilde{R})$ is continuously differentiable, we may apply the fundamental theorem of calculus along a path between $R$ and $\tilde{R}$, giving us

$$\log z^h(\tilde{R}) = \log z^h(R) + \int_0^{||\tilde{R} - R||} D_{\tilde{R} - R} \log z^h \left( R + \alpha \frac{\tilde{R} - R}{||\tilde{R} - R||} \right) d\alpha \tag{135}$$

$$\left| \log z^h(\tilde{R}) - \log z^h(R) \right| = ||\tilde{R} - R||2M \leq 2M\delta$$

where $D_{\tilde{R} - R}$ is the Frechet along the path between $R$ and $\tilde{R}$, and where we have used the bounds on $\log z^h(\tilde{R})$’s first derivative derived in the previous step. We conclude that $|z^h(\tilde{R})| \leq e^{26M}|z^h(R)| = e^{26M}z^h(\tilde{R})$, i.e. $z^h(R)$ bounds $z^h(\tilde{R})$ as desired.

**Integrability of retained income and its derivatives**

In this section we argue that—for any $\tilde{R} \in R + B_\delta(0)$, $\Delta, \bar{\Delta} \in \Delta — c^h(\tilde{R}) = \tilde{R}(z^h(\tilde{R}))$, $\tilde{D}_\Delta \tilde{R}(z^h(\tilde{R}))$, and $D^2_{\Delta\Delta} \tilde{R}(z^h(\tilde{R}))$ are bounded by $c^h_0$ and are integrable. WLOG, we show this in the case where $||\Delta|| = ||\bar{\Delta}|| = 1$. Since by Lemma 2 and the continuity of $R$ and $\tilde{R}$, we know that $c^h_0 = R(z^h(\tilde{R}))$, $c^h(\tilde{R}) = \tilde{R}(z^h(\tilde{R}))$, $D_\Delta \tilde{R}(z^h(\tilde{R}))$, and $D^2_{\Delta\Delta} \tilde{R}(z^h(\tilde{R}))$ are measurable, and since by Assumption 4, $c^h_0$ is integrable, the bounding is sufficient for integrability. To this end, a useful observation is that:

$$D_\Delta \log \tilde{R}(z^h(\tilde{R})) = \frac{d\log \tilde{R}(z^h(\tilde{R}))}{d\log z^h} D_\Delta \log z^h(\tilde{R}) + \Delta(\tilde{R}(z^h(\tilde{R}))) \frac{D_\Delta \log z^h(\tilde{R})}{R(z^h(\tilde{R}))}$$

$$D^2_{\Delta\Delta} \log \tilde{R}(z^h(\tilde{R})) = \frac{d^2\log \tilde{R}(z^h(\tilde{R}))}{d\log z^h^2} \left( \left( \frac{d\log \tilde{R}(z^h(\tilde{R}))}{d\log z^h} + 1 - \frac{d\log \tilde{R}(z^h(\tilde{R}))}{d\log z^h} \right) \right) D_\Delta \log z^h(\tilde{R}) + \Delta(\tilde{R}(z^h(\tilde{R}))) \frac{D_\Delta \log z^h(\tilde{R})}{R(z^h(\tilde{R}))} + \Delta(\tilde{R}(z^h(\tilde{R}))) \frac{D_\Delta \log z^h(\tilde{R})}{R(z^h(\tilde{R}))}$$

$$+ \Delta(\tilde{R}(z^h(\tilde{R}))) \frac{D_\Delta \log z^h(\tilde{R})}{R(z^h(\tilde{R}))}$$

$$+ \Delta(\tilde{R}(z^h(\tilde{R}))) \frac{D^2_{\Delta\Delta} \log z^h(\tilde{R})}{R(z^h(\tilde{R}))} \tag{136}$$

\[ \text{Note this is defined since } \tilde{R} - R \in \Delta \]
We have already provided—in earlier steps of the proof—uniform bounds (across $h \in \mathcal{H}$ and $\tilde{R} \in R + B_\delta(0)$) on
\[
\frac{d \log \tilde{R}(z^h(\tilde{R}))}{d \log z^h}, \quad \frac{d \log \tilde{R}'(z^h(\tilde{R}))}{d \log z^h}, \quad \frac{\Delta(z^h(\tilde{R}))}{R(z^h(\tilde{R}))}, \quad \frac{\Delta'(z^h(\tilde{R}))}{R'(z^h(\tilde{R}))}, \quad \frac{\Delta''(z^h(\tilde{R}))}{R''(z^h(\tilde{R}))}.
\]
\[
\hat{\Delta}'(z^h(\tilde{R})), \quad D_{\Delta} \log z^h(\tilde{R}), \quad D_{\Delta} \log z^h(\tilde{R}), \quad D_{\Delta} \log z^h(\tilde{R}), \quad \text{i.e. all of the terms above.}
\]

The same argument as in the previous step of this proof (connecting the bounds on $\log z^h(\tilde{R})$’s derivatives to those on $z^h(\tilde{R})$’s derivatives) then implies that the first two Frechet derivatives of $c^h(\tilde{R}) = \tilde{R}(z^h(\tilde{R}))$ are bounded across $\tilde{R} \in R + B_\delta(0)$ by $c_0^h$ times

\section{Proof of Lemma 5}

Here we show that within some neighborhood around $R$, $w^h \circ V^h(\tilde{R})$ is finite, $\mathcal{H}$-measurable, and has two continuous and $\mathcal{H}$-measurable Frechet derivative in $\mathcal{R}$.

To begin, fix a standard social objective $((w^h)_{h \in \mathcal{H}}, G)$ and take $\delta > 0$ small enough that Lemma 2 applies (so the definition of a standard social objective is well-defined), and so that conditions in the definition of a standard social objective hold.

Fixing any $h \in \mathcal{H}$, we now proceed to verify the conditions in the Lemma statement. First, finiteness is immediate from the definition of a standard social objective. Next, existence and continuity of derivatives: By Assumption 2 and the positivity of consumption and labor supply (Lemma 2), $u^h$’s first two derivatives in $(c, z)$ are continuous at all $c^h(\tilde{R}), z^h(\tilde{R})$ with $\tilde{R} = R + B_\delta(0)$. Since by Lemma 2 $z^h(\tilde{R})$ and $c^h(\tilde{R})$ and their first two derivatives are continuous in $\tilde{R} \in R + B_\delta(0)$, this is therefore also true of $V^h(\tilde{R}) = u^h(c^h(\tilde{R}), z^h(\tilde{R}))$. Since by the argument above $V^h(\tilde{R})$ is finite on this domain, the definition of a standard social objective implies that $w^h \circ V^h(\tilde{R})$ is twice-continuously differentiable in $\tilde{R} \in R + B_\delta(0)$. Finally, measurability of levels and derivatives: This is stated directly in the definition of a standard social objective.

\section{Proof of Lemma 5}

We complete the proof in several steps. We begin by deriving explicit formulas for the first two Frechet derivatives of $w^h \circ V^h(\tilde{R})$, and then argue that each is integrable.

\textbf{First derivative of welfare}

Throughout the proof we fix $\delta$ small enough that Lemmas 2 and 3 and the conditions in Definitions 1 and 3 of standard and regular social objectives apply.

By Lemma 4, we have that for all $h \in \mathcal{H}, \tilde{R} \in R + B_\delta(0)$ $w^h \circ V^h(\tilde{R})$ is twice-continuously Frechet differentiable. Since $w^h \circ V^h(\tilde{R}) = w^h \circ u^h(c^h(\tilde{R}), z^h(\tilde{R}))$, and by Lemma 2 $c^h(\tilde{R})$ and $z^h(\tilde{R})$ are twice-continuously Frechet differentiable, it is straightforward to compute the derivatives of $w^h \circ V^h$ (which exist by Assumption 2 and the definition of a standard social objective.)
For the first derivative, we have, for any $\Delta \in \Delta$,
\[
D_\Delta w^h \circ V^h(\tilde{R}) = (w^h \circ u^h)_{\epsilon^h} \left( \tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}) \right) \tilde{R}(z^h(\tilde{R})) D_\Delta \log \tilde{R}(z^h(\tilde{R}))
\]
\[
+ (w^h \circ u^h)_{z} \left( \tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}) \right) \tilde{R}(z^h(\tilde{R})) D_\Delta \log z^h(\tilde{R})
\]
\[
= (w^h \circ u^h)_{\epsilon^h} \left( \tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}) \right) \tilde{R}(z^h(\tilde{R})) \left[ \frac{d \log \tilde{R}(z^h(\tilde{R}))}{d \log z^h(\tilde{R})} D_\Delta \log z^h(\tilde{R}) + \frac{\Delta(z^h(\tilde{R}))}{R(z^h(\tilde{R}))} \right]
\]
\[
+ (w^h \circ u^h)_{z} \left( \tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}) \right) \tilde{R}(z^h(\tilde{R})) D_\Delta \log z^h(\tilde{R})
\]
\[
= (w^h \circ u^h)_{\epsilon^h} \left( \tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}) \right) \tilde{R}(z^h(\tilde{R})) \frac{\Delta(z^h(\tilde{R}))}{R(z^h(\tilde{R}))}
\]
\[\text{(137)}\]
where the cancellation is by the household’s first-order condition for labor supply (see \[102\] in the proof of Lemma 2). This generates an intuitive envelope expression.

**Second derivative of welfare**

To compute the second derivative of each household $h$’s contribution to welfare, we differentiate \[137\]. We have, for any $\Delta, \tilde{\Delta} \in \Delta$,
\[
D^2_{\Delta \tilde{\Delta}} w^h \circ V^h(\tilde{R}) = \left[ (w^h \circ u^h)_{cc} \left( \tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}) \right) \tilde{R}(z^h(\tilde{R})) \left( \frac{d \log \tilde{R}(z^h(\tilde{R}))}{d \log z^h(\tilde{R})} D_{\Delta} \log z^h(\tilde{R}) + \frac{\Delta(z^h(\tilde{R}))}{R(z^h(\tilde{R}))} \right) \right]
\]
\[
+ (w^h \circ u^h)_{z\epsilon^h} \left( \tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}) \right) \tilde{R}(z^h(\tilde{R})) D_{\Delta} \log z^h(\tilde{R})
\]
\[
= (w^h \circ u^h)_{cc} \left( \tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}) \right) \tilde{R}(z^h(\tilde{R})) \frac{\Delta(z^h(\tilde{R}))}{R(z^h(\tilde{R}))} \Delta(z^h(\tilde{R}))
\]
\[
+ (w^h \circ u^h)_{c} \left( \tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}) \right) \tilde{R}(z^h(\tilde{R})) D_{\Delta} \log z^h(\tilde{R})
\]
\[
= (w^h \circ u^h)_{cc} \left( \tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}) \right) \tilde{R}(z^h(\tilde{R})) \frac{\Delta(z^h(\tilde{R}))}{R(z^h(\tilde{R}))} \Delta(z^h(\tilde{R}))
\]
\[
+ (w^h \circ u^h)_{c} \left( \tilde{R}(z^h(\tilde{R})), z^h(\tilde{R}) \right) \tilde{R}(z^h(\tilde{R})) \frac{d \log \tilde{R}(z^h(\tilde{R}))}{d \log z^h(\tilde{R})} \frac{1}{\epsilon^h(\tilde{R})} D_{\Delta} \log z^h(\tilde{R})
\]
\[\text{(138)}\]

The second equality follows from that
\[
M^h(c^h, z^h) = -u^h(c^h, z^h)
\]
\[
\Rightarrow M^h(c^h, z^h) \frac{d}{d c^h} (w^h \circ u^h)(c^h, z^h) = -\frac{d}{d z^h} (w^h \circ u^h)(c^h, z^h)
\]
\[
\Rightarrow M^h(c^h, z^h) \cdot (w^h \circ u^h)_{c} + M^h \cdot (w^h \circ u^h)_{cc} = -(w^h \circ u^h)_{cc}
\]
\[\text{(139)}\]
where here \( z^h = z^h(\tilde{R}) \), \( c^h = \tilde{R}(z^h) \), elasticities and utility are are evaluated at \( \tilde{R} \), and we have using the definitions of \( \varepsilon^h \) and \( \eta^h \) and the fact that \( \tilde{R}'(z^h) = M^h(c^h, z^h) \) at the initial equilibrium.

**Boundedness and integrability of welfare and its derivatives**

Finally, we will argue that each of welfare \( w^h \circ V^h(\tilde{R}) \) and its first two Frechet derivatives are, as linear maps, bounded across all \( \tilde{R} \in R + B_\delta(0) \) by some linear combination of the functions \( b_n(h) \) from Definition 3 (of local regularity)\(^92\). As we have already argued that each is measurable, this also implies by dominated convergence that each is integrable.

The case of welfare itself (not its derivatives) is immediate from Definition 3. Next, consider the first derivative of welfare. By our expression (137) for the first derivative, the definition of \( ||\cdot|| \) and Definition 3, we have that \( |D_\Delta w^h \circ V^h(\tilde{R})| \leq ||\Delta||b_1(h) \) for all \( h \in H, \tilde{R} \in B_\delta(0) \), as desired. Finally, consider the second derivative of welfare. By our expression (138) for the second derivative, the definition of \( ||\cdot|| \), Definition 3, Assumptions 1 and 2, and the fact that any \( |D_\Delta \log z^h(\tilde{R})| \) is bounded by a constant times \( ||\Delta|| \) (by Lemma 3; see Footnote 60), we have that \( |D^2_\Delta w^h \circ V^h(\tilde{R})| \leq ||\Delta||(|\Delta|)(b_2(h) + \text{const} \cdot b_1(h)) \) for all \( h \in H, \tilde{R} \in B_\delta(0) \), as desired.

### E.6 Supporting details for Theorem 1

In this section we cover various details omitted from the proof of Theorem 1 in order to focus on the main points. Each supporting detail is referenced in the main proof.

#### E.6.1 Properties of aggregate revenue and welfare

By Lemmas 3 and 5 there exists \( \delta > 0 \) such that on \( R + B_\delta(0) \ni R \),

- \( z^h(\tilde{R}) \), \( c^h(\tilde{R}) \) are well-defined and have two Frechet derivatives in \( \tilde{R} \) that are continuous, \( H \)-integrable, and bounded by linear combinations of \( z_0^h \) and \( c_0^h \) (which are integrable by Assumption 1). By the linearity of differentiation, the same is true for tax revenue \( z^h(\tilde{R}) - c^h(\tilde{R}) \).
- \( w^h \circ V^h(\tilde{R}) \) is well-defined and has two Frechet derivatives in \( \tilde{R} \) that are continuous, \( H \)-integrable, and bounded by linear combinations the functions \( b_n(h) \) in Definition 3 (which are integrable by assumption).

Theorem 1 of \cite{Kammar2016} therefore implies that—for \( f^h(\tilde{R}) = z^h(\tilde{R}) - c^h(\tilde{R}) \) and for \( f^h(\tilde{R}) = w^h \circ V^h(\tilde{R}) - \int f^h(\tilde{R})d\mu \) is two-times Frechet differentiable on \( R + B_\delta(0) \) and that

\[
D \int f^h(\tilde{R})d\mu = \int Df^h(\tilde{R})d\mu \quad \text{and} \quad D^2 \int f^h(\tilde{R})d\mu = \int D^2 f^h(\tilde{R})d\mu. \quad (140)
\]

Moreover—by dominated convergence—\( \int f^h(\tilde{R})d\mu, D \int f^h(\tilde{R})d\mu, \) and \( D^2 \int f^h(\tilde{R})d\mu \) are continuous in \( \tilde{R} \) because \( f^h(\tilde{R}), Df^h(\tilde{R}), \) and \( D^2 f^h(\tilde{R}) \) are, by Lemma 2 and because

\(^92\)That is, for any \( \Delta, \tilde{\Delta} \in \Delta, w^h \circ V^h(\tilde{R}) \leq (a_0^h b_0(h) + a_1^h b_1(h) + a_2^h b_2(h)), D_\Delta w^h \circ V^h(\tilde{R}) \leq ||\Delta||(|\Delta|)(b_0(h) + b_1(h) + b_2(h)), \) and \( D^2_\Delta w^h \circ V^h(\tilde{R}) \leq ||\Delta||(|\Delta|)(a_0^h b_0(h) + a_1^h b_1(h) + a_2^h b_2(h)) \) for some constants \( a_j^h \in \mathbb{R} \).
each is dominated by an integrable function (a linear combination of \( z_0^h, c_0^h \), and the functions \( b_n(h) \)).

E.6.2 Measure-theoretic steps used to change variables

The following steps are implicit in the change of variables carried out in (54):

\[
\begin{align*}
\int \left[ (1 - R'(z_0^h)) z_0^h \left( \eta^h(R) \frac{\Delta(z_0^h)}{R(z_0^h)} + \varepsilon^h(R) \frac{\Delta'(z_0^h)}{R'(z_0^h)} \right) - \Delta(z_0^h) \right] d\mu \\
= \int_{\{ z_0^h \in \text{supp } g \}} \left[ (1 - R'(z_0^h)) z_0^h \left( \eta^h(R) \frac{\Delta(z_0^h)}{R(z_0^h)} + \varepsilon^h(R) \frac{\Delta'(z_0^h)}{R'(z_0^h)} \right) - \Delta(z_0^h) \right] d\mu \\
= \int_{\{ z_0^h \in \text{supp } g \}} \left[ (1 - R'(z)) z \left( \eta(z) \frac{\Delta(z)}{R(z)} + \varepsilon(z) \frac{\Delta'(z)}{R'(z)} \right) - \Delta(z) \right] d\mu_z \\
= \int_{\text{supp } g} \left[ (1 - R'(z)) z \left( \eta(z) \frac{\Delta(z)}{R(z)} + \varepsilon(z) \frac{\Delta'(z)}{R'(z)} \right) - \Delta(z) \right] g(z) dz
\end{align*}
\]

Above, the second line is by the absolute continuity of the integral and the fact that \( \mu(\{ z_0^h \in \text{supp } g \}) = 1 \). The third line follows from breaking the integral into each of its additively-separable components—each of which are integrable by bounding arguments in the first part of the proof of Lemma 3—then applying the definition of conditional expectation—and then adding these integrals back together them (by linearity of Lebesgue integration). The fourth line follows from changing variables and letting \( \mu_z \) denote the measure on \( \mathbb{R}_{\geq 0} \) induced by \( z_0^h(H) \). The fifth line is by the Radon-Nikodym / change of measure theorem and the definition of the density as a Radon-Nikodym derivative; the density exists by Assumption 5.

To see why this is sufficient, consider any continuous-in-\( \tilde{R} \) and integrable-over-\( h \) function \( \rho(\tilde{R}, h) \) which is bounded across all \( \tilde{R} \) by some integrable \( \overline{\rho(h)} \). For any sequence \( \tilde{R}_n \rightarrow \tilde{R} \), we therefore have—by continuity, the integrable bound, and dominated convergence—that

\[
\rho(\tilde{R}_n, h) \rightarrow \rho(\tilde{R}, h) \quad \Rightarrow \quad \int \rho(\tilde{R}_n, h) d\mu \rightarrow \int \rho(\tilde{R}, h) d\mu.
\]

See Chapter 3, Theorem 2.5, Shorack 2000.

See Chapter 3, Theorem 2.6, Shorack 2000.

See Chapter 4, Theorem 2.2, Shorack 2000.
E.6.3 Application of optimization-theoretic results

Recall the optimization problem (69):

\[ 0 \in \arg \max_{\Delta \in \Delta} F(\Delta) \quad \text{s.t.} \quad H(\Delta) \in \mathbb{R}_{\geq 0} \]

where

\[ F(\Delta) \equiv \begin{cases} \int u^h \circ u^h \left( (R + \Delta)(z^h(R + \Delta)), z^h(R + \Delta) \right) \, d\mu, & \text{if } \Delta \in B_\delta(0) \\ F(0) - 1 & \text{if } \Delta \notin B_\delta(0) \end{cases} \]

\[ H(\Delta) \equiv \begin{cases} \int \left[ z^h(R + \Delta) - (R + \Delta)(z^h(R + \Delta)) \right] \, d\mu - G, & \text{if } \Delta \in B_\delta(0) \\ 0, & \text{if } \Delta \notin B_\delta(0) \end{cases} \]

where \( \delta > 0 \) is small enough that \( F \) and \( H \) are well-defined and within \( B_\delta(0) \) have well-defined and continuous first and second Frechet derivatives (see Appendix E.6.1).

We wish to apply results from optimization theory on Banach spaces results to the problem above. In particular, we leverage Theorems 3.2 and 3.3 (part 2) of Maurer and Zowe [1979] in the special case of a one-dimensional constraint, restated below:

**Appendix Theorem 1.** Let \( X \) be a real Banach space, \( \bar{x} \) a point in \( X \), and \( F : X \to \mathbb{R} \) and \( H : X \to \mathbb{R} \) functions whose first and second Frechet derivatives exist at \( \bar{x} \). Suppose that

\[ \bar{x} \in \arg \max_{H(x) \geq 0} F(x), \quad (144) \]

\( H(\bar{x}) = 0 \), and for some \( h \in X \), \( D_h H(\bar{x}) \neq 0 \) (\( \bar{x} \) is optimal and full-rank / regular, and \( H \) binds).\[^{97}\]

Then there exists \( \kappa \in \mathbb{R}_{\geq 0} \) such that \( DF(\bar{x}) + \kappa DH(\bar{x}) = 0 \) and, for all non-zero \( d \in X \) satisfying \( D_d H(\bar{x}) = 0 \), \( D^2_{d,d} F(\bar{x}) + \kappa D^2_{d,d} H(\bar{x}) \leq 0 \).

In order to apply Appendix Theorem 1 we must verify that the optimization problem (69) satisfies several conditions. It is immediate from the setup above that:

- \( F \) is a functional defined on a real Banach space, \( (\Delta, ||\cdot||)_H \).\[^{98}\]
- \( G \) is a map from \( \Delta \) into \( \mathbb{R} \), a real Banach space (w.r.t. the standard norm).
- The feasible range for \( G \) is \( \mathbb{R}_{\geq 0} \).
- \( F \) restricted to \( G^{-1}(\mathbb{R}_{\geq 0}) \) achieves a local (indeed, global) maximum at \( 0 \).
- The first and second Frechet derivatives of \( F \) and \( G \) exist at \( 0 \).

In order to reach the result stated in the main proof of Theorem 1, it remains to verify that \( H(0) = 0 \), i.e. the revenue constraint binds. To see this suppose not. Since \( H \) is continuous local to \( 0 \), and \( F \) is differentiable local to \( 0 \), it suffices to show that for any \( \epsilon > 0 \), there exists \( \Delta \in \Delta \) with \( F(\epsilon \Delta) > F(0) \). To see this, define \( \Delta = R \) and note that \( R(z^h) > 0 \) for all \( h \in \mathcal{H} \) (this follows from the fact shown in the proof of Lemma 2 that...
$R(z) > 0$ for all $z > 0$ and from Assumption 3. By Assumption 2, $V^h(R + \epsilon \Delta) > w^h \left( c_0^h + \epsilon R(z_0^h) \right) > V^h(R)$. So by Definition 1, $w^h \cdot V^h(R + \epsilon \Delta) > w^h \circ V^h(R)$ for a positive measure of households $h$; so $F(\epsilon \Delta) > F(0)$.

### E.6.4 Zero-rank case of (69)

Consider the optimization problem (69) and suppose that $D H(0) = 0$, i.e. $D \Delta(0) = 0$ for all $\Delta \in \Delta$. We claim in this case (ABC) holds with equality for all $z \in \text{supp} \, g$, so the theorem holds.

The argument is identical to that of the section “First-order condition” in the main proof of Theorem 1 except that (a) one may start at (70), which now holds with equality, and (b) the argument in the next paragraph implies $\psi(z) = 0$, because of (a).

### E.6.5 Example of first-order tax deviation

The main proof of Theorem 1 uses the existence of a weakly positive function $\Delta(z)$ that is strictly positive on a non-zero-measure sub-interval of $[z, \overline{z}]$, zero outside of $[z, \overline{z}]$, and is contained in $\Delta$. We now give an example of such a function.

We begin by defining the “deviation function”, for any $\bar{z} > 0$, $\delta \in (0, \bar{z})$, by

$$\tilde{\Delta}(z; \bar{z}, \delta) \equiv \begin{cases} \left( \frac{z-(\bar{z}-\delta)}{\delta} \right)^5 \left( \frac{z+(\bar{z}+\delta)}{\delta} \right)^5 & \text{if } z \in B_{\delta}(\bar{z}) \\ 0 & \text{otherwise.} \end{cases} \quad (145)$$

It is easy to verify that $\tilde{\Delta}(\cdot; \bar{z}, \delta)$ has four continuous and bounded derivatives, is strictly positive in $B_{\delta}(\bar{z})$ and zero elsewhere, and has $\tilde{\Delta}(z; \bar{z}, \delta) \leq \tilde{\Delta}(\bar{z}; \bar{z}, \delta) = 1$. Moreover note that $R(z)$ and $R'(z)$ are bound strictly above zero in $\overline{B_{\delta}(\bar{z})}$ because $R(z), R'(z) > 0$ at all $z > 0$—as argued in the proof of Lemma 2—and $R$ and $R'$ are continuous by Assumption 1. Together, these observations imply $\tilde{\Delta}(\cdot; \bar{z}, \delta) \in \Delta$.

To obtain the desired deviation $\Delta$, consider $\Delta \equiv \tilde{\Delta}(\cdot; \bar{z}, \bar{z})$.

### E.6.6 Example of second-order tax deviation

The main proof of Theorem 1 uses the existence of, for any $k > 0$, $r > 0$, $\bar{z} \in \text{supp} \, g$, a tax change $\Delta$ that is in $\Delta$, is zero outside of the interval $B_r(\bar{z})$, and satisfies $\int_{\bar{z}-r}^{\bar{z}+r} \Delta'(\bar{z})^2 \, dz > k \int_{\bar{z}-r}^{\bar{z}+r} \Delta(\bar{z})^2 \, dz$. Here the idea is to take $\Delta(\cdot; \bar{z}, r, k)$ to be a sufficiently narrow “bump” centered at $\bar{z}$. We now provide an example of some such function.

To construct the example, we use the “deviation function” defined in Appendix E.6.5. Specifically, we take $\Delta_N \equiv \Delta(\cdot; \frac{\bar{z}+r}{2}, r/N)$ for some $N \geq 1$. We have already established in Appendix E.6.5 that $\Delta_N \in \Delta$ and $\Delta_N$ is zero outside of the interval $B_r(\bar{z})$. To see that there exists $N$ for which we obtain

$$\int_{\bar{z}-r}^{\bar{z}+r} \Delta_N'(\bar{z})^2 \, dz > k \int_{\bar{z}-r}^{\bar{z}+r} \Delta_N(\bar{z})^2 \, dz, \quad (146)$$

note that as $N \to \infty$, the RHS converges to 0, whereas the LHS diverges to $\infty$.

---

99\text{It is immediate from the definition of } \Delta \text{ that } R \in \Delta.
E.7 Supporting details for Theorem 2

In this section we cover various details omitted from the proof of Theorem 2 in order to focus on the main points. Each supporting detail is referenced in the main proof.

E.7.1 Properties of \( \hat{\lambda}(z) \) and \( \hat{\lambda}^h \)

We make several observations about \( \hat{\lambda}(z) \) and \( \hat{\lambda}^h \).

First, \( \hat{\lambda}(z) \) is continuously differentiable on \( \text{supp} \, g \). This is immediate from the definition of \( \Pi_{ABC}(z) \) and Assumptions 2, 3, and 6.

Second, there exists \( M > 0 \) such that for all \( z \in \text{supp} \, g \), \( z|\hat{\lambda}(z)| \leq M|\lambda(z)| \). This is immediate from the conditions on \( \Pi_{ABC}(z) \) in the statement of the theorem.

Third, there exist \( \lambda_c, \lambda_z \in \mathbb{R}_{>0} \) such that for all \( z \in \text{supp} \, g \), \( R(z) \hat{\lambda}(z) \leq \lambda_c R(z) + \lambda_z z \). This can be seen by considering the definition of \( \Pi_{ABC}(z) \) and making the following observations:

- \( R'(z)z \) is bounded across all \( z \in \mathbb{R}_{\geq 0} \) by \( B^R R(z) \), by Assumption 2.
- \( \eta(z), \varepsilon(z) \) are bounded across \( \text{supp} \, g \) by a constant by Assumption 4.
- \( \alpha(z), \frac{d \log \varepsilon(z)}{d \log z} = z \varepsilon'(z), \) and \( \frac{R(z)}{R'(z)} \) are bounded across \( \text{supp} \, g \) by constants, by Assumption 6.
- For all \( z \in \text{supp} \, g \), \( R(z) \left| \frac{d}{d \log z} \left( \frac{1-R'(z)}{R'(z)} \right) \right| = R(z) \left( \left| \frac{R''(z)}{R'(z)} \right| + \left| \frac{1-R'(z)}{R'(z)} \right| \right) \leq B^R (1 + 1) R(z) + B^R k z \) for some constant \( k \) by Assumption 1 and the boundedness of \( \frac{R(z)}{R'(z)z} \).

Fourth, \( \hat{\lambda}^h \) is measurable in \( \mathcal{H} \). This follows from that (a) \( \varepsilon^h(R) \) is measurable (see the second-to-last section of the proof of Lemma 2), (b) within \( \text{supp} \, g \), \( p(z; \epsilon) \) is continuous in \( z \) and strictly positive by assumption, (c) \( \lambda_0 \) is measurable by Assumption 3, (d) \( \hat{\lambda}(z) \) is piece-wise continuous, and (e) \( \text{supp} \, g \) is measurable by Lemma 3 in Appendix E.8.

Fifth, \( R(z_0^h) \hat{\lambda}^h \) is integrable. We will show this by an application of Fatou’s Lemma.\( ^{100} \)

Define, for \( n \in \mathbb{N} \), \( \hat{\lambda}_n^h \equiv \hat{\lambda}(z_0^h) \mathbb{I}_{v^h(R) \leq \epsilon} \min \left[ \frac{1}{p_{z_0^h; \epsilon}}, \right] \), and note that \( \hat{\lambda}_n^h \geq 0 \) and that \( \hat{\lambda}_n^h \to \hat{\lambda}^h \) pointwise. Moreover, \( R(z_0^h) \hat{\lambda}_n^h \) is integrable since is measurable (by a similar argument to that used for \( R(z_0^h) \hat{\lambda}^h \)) and bounded by \( n R(z_0^h) \hat{\lambda}(z_0^h) \), which recall is itself bounded by \( n \lambda_c R(z_0^h) + n \lambda_z z_0^h \) (which is integrable by Assumption 4). Further, note that

\[
\mathbb{E}[R(z_0^h) \hat{\lambda}_n^h] = \mathbb{E}\left[ \mathbb{E}[R(z_0^h) \hat{\lambda}_n^h | z_0^h] \right] = \mathbb{E}\left[ \mathbb{E}\left[ R(z_0^h) \hat{\lambda}(z_0^h) \mathbb{I}_{v^h(R) \leq \epsilon} \min \left[ \frac{1}{p_{z_0^h; \epsilon}}, \right] \right] \right] \\
= \mathbb{E}\left[ R(z_0^h) \hat{\lambda}(z_0^h) \mathbb{I}_{v^h(R) \leq \epsilon} \min \left[ \frac{1}{p_{z_0^h; \epsilon}}, \right] \right] \\
= \mathbb{E}\left[ R(z_0^h) \hat{\lambda}(z_0^h) \mathbb{I}_{v^h(R) \leq \epsilon} \min \left[ \frac{1}{p_{z_0^h; \epsilon}}, \right] \right] \\
\leq \mathbb{E}\left[ R(z_0^h) \hat{\lambda}(z_0^h) \right] (174)\]

\(^{100}\)Fatou’s Lemma is a standard result in measure theory. In words, it says that, for non-negative real-valued random variables, the lim-inf of an expectation is less than the expectation of a lim-inf.
where here we have used $z^h_0$’s measurability to take the conditional expectation, then used the tower property of conditional expectations, and then taken advantage of $1_{x^h(R) \leq \varepsilon}$’s integrability to pull out $z^h_0$-measurable terms from the conditional expectation. Since $R(z^h_0)\hat{\lambda}^h$, and $R(z^h_0)\hat{\lambda}^h_n$ are all non-negative, we have by Fatou’s Lemma (and then the bounds above) that

$$\int R(z^h_0)\hat{\lambda}^h d\mu = \int \liminf_{n \to \infty} R(z^h_0)\hat{\lambda}^h_n d\mu \leq \liminf_{n \to \infty} \int R(z^h_0)\hat{\lambda}^h_n d\mu \leq E \left[ 1_{z^h_0 \in \text{supp} g} R(z^h_0)\hat{\lambda}(z^h_0) \right] < \infty$$

(148)

In particular, $R(z^h_0)\hat{\lambda}^h$ is integrable.

E.7.2 Conditional expectations related to $\hat{\lambda}^h$

We first argue that several conditional expectations of interest—namely those of $R(z^h_0)\hat{\lambda}^h$, $R(z^h_0)\hat{\lambda}^h\eta^h(R)$, $R(z^h_0)\hat{\lambda}^h\varepsilon^h(R)$, and $R(z^h_0)\hat{\lambda}^h \frac{\eta^h(R)^2}{\varepsilon^h(R)}$—conditional on income—exist. Second, we show how these moments can be related to conditional expectations of $\hat{\lambda}^h\eta^h(R)$, $\varepsilon^h(R)$ and $\frac{\eta^h(R)^2}{\varepsilon^h(R)}$. Finally, we argue that each conditional expectation is continuous in income and that the conditional expectation of $R(z^h_0)\hat{\lambda}^h\eta^h(R)$ is continuously differentiable in income. These arguments rely on facts shown in Appendix E.7.1.

To begin, we argue that the following conditional expectation functions exist:

$$E \left[ R(z^h_0)\hat{\lambda}^h \mid z^h_0 \right], \ E \left[ R(z^h_0)\hat{\lambda}^h \eta^h(R) \mid z^h_0 \right], \ E \left[ R(z^h_0)\hat{\lambda}^h \varepsilon^h(R) \mid z^h_0 \right], \ E \left[ R(z^h_0)\hat{\lambda}^h \frac{\eta^h(R)^2}{\varepsilon^h(R)} \mid z^h_0 \right].$$

(149)

Since by Assumption \[3\] $z^h_0$ is measurable, this follows so long as $R(z^h_0)\hat{\lambda}^h$, $R(z^h_0)\hat{\lambda}^h\eta^h(R)$, $R(z^h_0)\hat{\lambda}^h\varepsilon^h(R)$, and $R(z^h_0)\hat{\lambda}^h \frac{\eta^h(R)^2}{\varepsilon^h(R)}$ are each integrable. Indeed, each is measurable by the observations (above) that $\hat{\lambda}^h$ is measurable and (see the second-to-last step of the proof of Lemma \[2\]) that elasticities are measurable; and each is bounded by an integrable function, by Assumption \[4\] and our observation above that $R(z^h_0)\hat{\lambda}^h$ is integrable.

Next we observe that, for $x^h = 1, \eta^h(R), \varepsilon^h(R), \frac{\eta^h(R)^2}{\varepsilon^h(R)}$; for any conditional expectations

\cite[See, e.g., Theorems 34.3 and 34.4 of Billingsley 2008.](#)
\[
\mathbb{E} \left[ R(z_0^h) \hat{\lambda}^h x^h \mid z_0^h \right] \quad \text{and} \quad \mathbb{E} \left[ x^h \mid z_0^h, \mathbf{1}_{e^h(R) \leq \epsilon} \right]
\]
and with probability one,

\[
\mathbb{E} \left[ R(z_0^h) \hat{\lambda}^h x^h \mid z_0^h \right] = \mathbb{E} \left[ R(z_0^h) \hat{\lambda}(z_0^h) \frac{\mathbf{1}_{e^h(R) \leq \epsilon}}{p \leq (z_0^h, \epsilon)} x^h \mid z_0^h \right]
\]

\[
= \mathbb{E} \left[ R(z_0^h) \hat{\lambda}(z_0^h) \frac{\mathbf{1}_{e^h(R) \leq \epsilon}}{p \leq (z_0^h, \epsilon)} x^h \mid z_0^h, \mathbf{1}_{e^h(R) \leq \epsilon} \right] \mid z_0^h \]

\[
= \mathbb{E} \left[ R(z_0^h) \hat{\lambda}(z_0^h) \frac{\mathbf{1}_{e^h(R) \leq \epsilon}}{p \leq (z_0^h, \epsilon)} \mathbb{E}[x^h \mid z_0^h, \mathbf{1}_{e^h(R) \leq \epsilon}] \mid z_0^h \right]
\]

\[
= \mathbb{E} \left[ R(z_0^h) \hat{\lambda}(z_0^h) \frac{\mathbf{1}_{e^h(R) \leq \epsilon}}{p \leq (z_0^h, \epsilon)} \mathbb{E}[x^h \mid z_0^h, \mathbf{1}_{e^h(R) \leq \epsilon}] \mid z_0^h \right]
\]

\[
= \mathbb{E} \left[ R(z_0^h) \hat{\lambda}(z_0^h) \frac{\mathbf{1}_{e^h(R) \leq \epsilon}}{p \leq (z_0^h, \epsilon)} \mathbb{E}[x^h \mid z_0^h, \mathbf{1}_{e^h(R) \leq \epsilon}] \mid z_0^h \right]
\]

\[
= R(z_0^h) \hat{\lambda}(z_0^h) \frac{\mathbf{1}_{e^h(R) \leq \epsilon}}{p \leq (z_0^h, \epsilon)} \mathbb{E}[x^h \mid z_0^h, \mathbf{1}_{e^h(R) \leq \epsilon}] \mid z_0^h
\]

\[
= R(z_0^h) \hat{\lambda}(z_0^h) x_{\leq (z_0^h, \epsilon)}
\]

\[(150)\]

where recall \(x_{\leq (z_0^h, \epsilon)} = \mathbb{E}[x^h \mid z_0^h = z, e^h(R) \leq \epsilon] \) is as defined in Assumption 6 and where for \(x^h = 1\), \(x_{\leq (z_0^h, \epsilon)} \) simply denotes 1. Above, the second equality holds (with probability one) by the tower property for any inner conditional expectation. The third and sixth equalities hold since the pulled-out terms are measurable with respect to \((z_0^h, \mathbf{1}_{e^h(R)_\leq \epsilon})\) and \(z_0^h\) in the respective cases. The third line also uses that \(\mathbb{E}[x^h \mid z_0^h, \mathbf{1}_{e^h(R)_\leq \epsilon}] \) exists, since \(x^h\) is integrable and \(z_0^h\) and \(\mathbf{1}_{e^h(R)_\leq \epsilon}\) are measurable; the sixth uses that \(\mathbb{E}[\mathbf{1}_{e^h(R)_\leq \epsilon} \mid z_0^h] \) exists, since \(e^h(R)\) and \(z_0^h\) are measurable and any indicator is bounded. The fourth equality is definitional and the fifth is immediate. The cancellations on the second-to-last line are with probability one, since \(p \leq (z_0^h, \epsilon)\) is a conditional expectation of \(\mathbf{1}_{e^h(R)_\leq \epsilon}\) on \(z_0^h\). The seventh equality is because all conditional expectations of the same variables are equal with probability one.

Since—for any choice of conditional expectation (conditional expectations are only unique up to differences on measure zero sets)—\(\mathbb{E}[R(z_0^h) \hat{\lambda}^h x^h \mid z_0^h]\) and \(R(z_0^h) \hat{\lambda}(z_0^h) x_{\leq (z_0^h, \epsilon)}\) coincide with probability one, and since \(R(z_0^h) \hat{\lambda}(z_0^h) x_{\leq (z_0^h, \epsilon)}\) is \(z_0^h\)-measurable, it follows from the definition of conditional expectation that \(R(z_0^h) \hat{\lambda}(z_0^h) x_{\leq (z_0^h, \epsilon)}\) is a conditional expectation for \(R(z_0^h) \hat{\lambda}^h x^h\) conditional on \(z_0^h\). For the remainder of the proof, we will work with this particular choice of conditional expectation.

Another consequence of our observations is that, for \(x^h = e^h(R), \eta^h(R), \eta^h(R)^2, \eta^h(R)^3, \eta^h(R)^4\), there exist functions \((\hat{\lambda}x) : \text{supp } g \to \mathbb{R} \text{— namely } (\hat{\lambda}x)(z) = \hat{\lambda}(z)x_{\leq (z, \epsilon)}\) that are equal to \(\mathbb{E}[R(z_0^h) \hat{\lambda}^h x^h \mid z_0^h = z] \) for all \(z \in \text{supp } g\). Moreover, the continuous differentiability of \(\hat{\lambda}(z)\) (shown above) and Assumption 6 imply that \((\hat{\lambda}e)(z)\) and \((\hat{\lambda} z^2)(z)\)—as well as, by Ass-

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102The latter conditional expectation exists since each \(x^h\) is integrable (by observations above and Assumption 4) and since \(\mathbf{1}_{e^h(R)_\leq \epsilon}\) is measurable since \(e^h(R)\) is.

103See, e.g., Theorem 34.4 of Billingsley [2008].

104The inner conditional expectation exists by the bounding arguments above and since \(\mathbf{1}_{e^h(R)_\leq \epsilon}\) is measurable.

105See, e.g. Theorem 34.3 of Billingsley [2008].
E.7.3 Properties of \((\hat{\lambda}_\gamma)(z)\) and \(\hat{\gamma}^h\)

We make several observations about \((\hat{\lambda}_\gamma)(z)\) and \(\hat{\gamma}^h\).

First, \((\hat{\lambda}_\gamma)(z)\) is continuous on supp \(g\). This follows from \(\hat{\lambda}(z)\)'s continuity, Lemma 6 and Assumptions 2, 4, and 6.

By Assumptions 4 and 6 and by our observations in Appendices E.7.1 and E.7.2,

- \((\hat{\lambda}_\gamma)(z)\) is continuous on supp \(g\).

By Lemma 6, \(\Psi_0(z)\) is bounded on supp \(g\) by a linear combination of \(z\) and \(R(z)\).

\[
\frac{d\log R(z)}{d\log z} \leq B^R \text{ by Assumption 1}
\]

By our observations in Appendix E.7.2, \(R(z)(\hat{\lambda}_\gamma)(z) = R(z)(\hat{\lambda}(z)x_\leq(z; \epsilon))\) is—on supp \(g\)—bounded by a linear combination of \(z\) and \(R(z)\) since—for \(x = \eta, \xi, \eta^2 - x_\geq(z; \epsilon)\) is bounded (by Assumption 4) and \(R(z)\hat{\lambda}(z) \leq \lambda_z R(z) + \lambda_z z\).

By Assumption 6, \(\alpha(z)\) is bounded in supp \(g\).

By Assumptions 4 and 6 and by our observations in Appendices E.7.1 and E.7.2, there exist constants \(M, M', M''\) such that for all \(z \in \text{supp} g\)

\[
R(z)(\lambda_\eta')(z) = R(z) \frac{d}{dz} (\lambda(z)\eta_\leq(z; \epsilon)) = R(z) \left( z\hat{\lambda}(z)\eta_\leq(z; \epsilon) + \hat{\lambda}(z)z\eta_\leq(z; \epsilon) \right)
\]

\[
\leq R(z) (M\hat{\lambda}(z)\eta_\leq(z; \epsilon) + M'\hat{\lambda}(z)) \leq M''R(z)\hat{\lambda}(z)
\]

which is integrable.

Third, \(\hat{\gamma}^h\) is measurable in \(h\). This follows from that (a) as shown above and noted / proved in Appendix E.7.1, \((\hat{\lambda}_\gamma)(z^h_0), \hat{\lambda}^h, \epsilon^h(R),\) and \(p(z^h_0; \epsilon)\) are all measurable functions of \(h\); (b) by Lemma 9 in Appendix E.8, supp \(g\) is a measurable set; and (c) for all \(h\) with \(z^h_0 \in \text{supp} g\) and \(\epsilon^h(R) \leq \epsilon, \hat{\lambda}^h > 0\) and \(p(z^h_0; \epsilon) > 0\) (by Assumption 6).

Fourth \(R(z^h_0)\hat{\lambda}^h\hat{\gamma}^h\) is integrable. To see this first note that \(R(z^h_0)(\hat{\lambda}_\gamma)(z^h_0)\) is integrable, which follows from (a) the continuity of \(R\) (Assumption 1); (b) the measurability of \(z^h_0, \hat{\lambda}^h,\) and \(\hat{\gamma}^h\) (Assumption 3, Appendix E.7.1 above); and (c) the fact that \(|R(z^h_0)(\hat{\lambda}_\gamma)(z^h_0)| \leq \lambda_\gamma cR(z^h_0) + \lambda_\gamma z^h_0\) for all \(h \in \mathcal{H}\), where note the RHS is integrable by Assumption 4. One may then apply the same Fatou’s Lemma argument used in Appendix E.7.2.

Fifth, we claim \(R(z^h_0)(\hat{\lambda}_\gamma)(z^h_0)\) is a conditional expectation for \(R(z^h_0)\hat{\lambda}^h\hat{\gamma}^h\) given \(z^h_0\). To see this, note that since \(z^h_0\) is measurable, we may consider the conditional expectation

\[
\mathbb{E} \left[ R(z^h_0)(\hat{\lambda}_\gamma)(z^h_0)|z^h_0 \right] = E \left[ R(z^h_0)(\hat{\lambda}_\gamma)(z^h_0) \mathbb{1}_{p^h(R) \leq \epsilon} \frac{z^h_0}{p(z^h_0; \epsilon)} \right]
\]

\[
= R(z^h_0)(\hat{\lambda}_\gamma)(z^h_0) \frac{1}{p(z^h_0; \epsilon)} \mathbb{E} \left[ \mathbb{1}_{p^h(R) \leq \epsilon} \frac{z^h_0}{p(z^h_0; \epsilon)} \right]
\]

(152)
where the second equality holds with probability one (same logic as in \[150\]). By the same logic as in Appendix E.7.2, \(R(\tilde{z}_0^h)(\bar{X}_t^\gamma)(z_0^h)\) is a conditional expectation for \(R(z_0^h)\tilde{X}_t^h\) given \(z_0^h\). For the remainder of the proof, we work with this particular choice of conditional expectation.

### E.7.4 Characterization of \(\hat{c}^h(u)\)

The main proof of Theorem 2 relies on a characterization of the compensating-consumption function \(\hat{c}^h(u) \equiv u^h(\cdot, z_0^h)^{-1}(u)\). While useful, this characterization is very tedious, and so we (further) relegate its proof to Appendix E.9. Here, we simply state the result, i.e., Lemma 11.

There exists \(\delta > 0\) small enough that the function

\[
\hat{c}^h(u) \equiv u^h(\cdot, z_0^h)^{-1}(u)
\]

is, for all \(h \in \mathcal{H}\), well-defined and strictly positive when \(u = V^h(\tilde{R})\) for some \(\tilde{R} \in R + B_\delta(0)\); moreover, \(\hat{c}^h(V^h(\tilde{R}))\) is \(\mathcal{H}\)-measurable. Further, there exists \(\tilde{m} > 0\) such that for all \(h \in \mathcal{H}, \tilde{R} \in R + B_\delta(0)\) and—for all real-valued functions \(\phi^h\) that are defined and twice differentiable in a neighborhood around \(V^h(\tilde{R})\) and satisfy \(\hat{c}^{\prime}h(V^h(\tilde{R})) > 0\)—we have

\[
\left| \log \hat{c}^h \left(V^h(\tilde{R})\right) - \log \hat{c}^h \left(\tilde{R}\right) \right| \leq \tilde{m}
\]

\[
\left| \log \left[ (\phi^h \circ u^h)_c \left(\hat{c}^h \left(V^h(\tilde{R}), z_0^h\right)\right) \right] - \log \left[ (\phi^h \circ u^h)_c \left(\hat{c}^h(\tilde{R}), z_0^h\right) \right] \right| \leq \tilde{m}
\]

\[
\left| \frac{d \log}{d \log c} \phi^h \circ u^h_c \left(\hat{c}^h \left(V^h(\tilde{R}), z_0^h\right)\right) - \frac{d \log}{d \log c} \phi^h \circ u^h_c \left(\hat{c}^h(\tilde{R}), z_0^h\right) \right| \leq \tilde{m}.
\]

### E.7.5 Properties of \(w^h(\cdot)\)

In this section, we establish that \(w^h\) is well-defined and that \(((w^h)_{h \in \mathcal{H}}, G)\) is a standard, regular social objective.

**Proof \(w^h\) is well-defined**

First, we argue \(w^h\) is well-defined. First, all terms in the integrand within the definition of \(w^h\) are defined for \([V_0^h, u]\) because—as \(V^h(R + B_\delta(0))\) is convex—any \(\tilde{u} \in [V_0^h, u]\) is equal to \(V^h(\tilde{R})\) for some \(\tilde{R} \in V^h(R + B_\delta(0))\), which by Lemma 11 implies \(\hat{c}^h(\tilde{u})\) is defined and strictly positive; \(u^h_c(\hat{c}^h(\tilde{u}), z_0^h)\) is therefore defined and strictly positive by Assumption 2. Next, note that this integrand is continuous in \(\tilde{u}\), by Assumption 2 and the fact that \(\hat{c}^h(\tilde{u})\) is continuous (which follows from that \(\hat{c}^h(u) = u^h(\cdot, z_0^h)^{-1}(u)\) and—since \(\hat{c}^h(u) > 0\) with \(u^h\) is is locally differentiable with \(u^h > 0\), by Assumption 2). Since the

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106 The fact that \(\hat{c}^h(V^h(\tilde{R})) > 0\) implies that \(u^h\) twice differentiable and has strictly positive first consumption derivative at all inputs where evaluated above, by Assumption 2, \(\phi^h\) is twice differentiable by assumption. Finally, since \(\phi^{\prime}h(V^h(\tilde{R})) > 0\) by assumption. Together, these observations imply all derivatives and logs used in the Lemma statement are well-defined.

107 This follows from that \(R + B_\delta(0)\) is convex and—by Assumption 2 and Lemma 2—plus the fact that within \(R + B_\delta(0), \hat{c}^h(\tilde{R}), z^h(\tilde{R})\) are strictly positive (this follows from integrating the bounds on their logs from Lemma 3—\(V^h(\tilde{R}) = u^h(\hat{c}^h(\tilde{R}), z^h(\tilde{R}))\) is continuous in \(\tilde{R}\).
integrable is continuous on a closed, bounded domain, the integral in the definition of \( w^h \) exists (and is finite).

**Proof of standard-ness**

We now argue that \( (w^h)_{h \in \mathcal{H}}, G \) is a *standard social objective* in the sense of Definition [1].

First, for all \( h \in \mathcal{H} \), \( w^h \) is a function from \( \text{Im}(u^h) \) to \( \mathbb{R} \cup \{-\infty\} \). Moreover, the definition of \( w^h \) and the argument for its well-definition in the previous step imply that \( w^h \) is always finite-valued.

Second, we claim that for all \( h \in \mathcal{H} \) and \( \tilde{R} \in R + B_\delta(0) \), \( w^h(u) \) is twice-continuously differentiable on the domain \( u \in V^h(R + B_\delta(0)) \). To see that \( w^h(u) \) is once-continuously differentiable on this domain follows from the fundamental theorem of calculus and the observation—established in the previous step of the proof—that the integrand in the definition of \( w^h \) is continuous in \( \tilde{u} \); indeed,

\[
    w^{h'}(u) = \hat{\lambda}^h \frac{\Phi(\gamma^h(\log \hat{\epsilon}^h(u) - \log c_0^h))}{u^h(\hat{\epsilon}^h(u), z_0^h)}.
\]

To establish that \( w^{h'}(u) \) is itself continuously differentiable, it suffices—given Assumption [2] and the observation in the proof of \( w^h \)'s well-defined-ness \( \hat{\epsilon}^h(u) > 0 \) for \( u \in V^h(R + B_\delta(0)) \)—to show that \( \hat{\epsilon}^h(u) \) is continuously differentiable in \( u \). This in turn follows from the implicit function theorem, \( u^h(\hat{\epsilon}^h(u), z_0^h) \)'s continuous differentiability (from Assumption [2] and \( \hat{\epsilon}^h(u) > 0 \)), and that \( u^h(\hat{\epsilon}^h(u), z_0^h) > 0 \).

Third, we claim that for all \( \tilde{R} \in R + B_\delta(0) \), \( w^h(u) \) is weakly increasing in \( u \) for all \( h \in \mathcal{H} \) and strictly increasing for a finite measure of \( h \in \mathcal{H} \). From (155) it is clear that \( w^h(u) \) is weakly and moreover strictly increasing for all \( h \in \mathcal{H} \) with \( \hat{\lambda}^h \geq 0 \) and moreover \( \hat{\lambda}^h > 0 \), respectively. It therefore suffices to show \( \hat{\lambda}^h \geq 0 \) for all \( h \in \mathcal{H} \) and \( \hat{\lambda}^h > 0 \) for a finite measure of \( h \). This follows from the definition of \( \hat{\lambda}^h \) and the observation in Appendix [E.7.1] that \( \hat{\lambda}(z) > 0 \) for all \( z \in \text{supp } g \).

In order to show that \( (w^h)_{h \in \mathcal{H}}, G \) is a standard social objective, it remains to show that for all \( \tilde{R} \in R + B_\delta(0) \) and all \( \Delta, \hat{\Delta} \in \Delta, w^h(V^h(\tilde{R})), D_{\Delta}w^h(V^h(\tilde{R})), \) and \( D^2_{\Delta\hat{\Delta}}w^h(V^h(\tilde{R})) \) are measurable in \( h \). We begin by computing the three terms explicitly, using \( w^h \)'s twice-continuous differentiability and using the fact that—since \( \hat{\epsilon}^h(\tilde{R}) \) and \( z^h(\tilde{R}) \) are twice-continuously differentiable by Lemma [2] and strictly positive by integrating the bounds on their derivatives in Lemma [3] and since \( u^h \) is twice-continuously differentiable on \( \mathbb{R}_{>0} \times \mathbb{R}_{>0} \) by [2] \( V^h(\tilde{R}) = u^h(\hat{\epsilon}^h(\tilde{R}), z^h(\tilde{R})) \) is twice-continuously Frechet.
\[
    \chi^h \left( V^h(\tilde{R}) \right) = \hat{\chi}^h \left( \frac{e^{\Phi(\gamma^h (\log \hat{c}^h(\tilde{u}) - \log \hat{c}^0))}}{u_c^h(\hat{c}^h(\tilde{u}), z_0^h)} \right) d\tilde{u} \\
    D\Delta w^h(V^h(\tilde{R})) = \hat{\chi}^h \frac{e^{\Phi(\gamma^h (\log \hat{c}^h(V^h(\tilde{R}) - \log \hat{c}^0))}}{u_c^h(\hat{c}^h(V^h(\tilde{R})), z_0^h)} D\Delta V^h(\tilde{R}) \\
    D^2 \Delta \Delta w^h(V^h(\tilde{R})) = \hat{\chi}^h \frac{e^{\Phi(\gamma^h (\log \hat{c}^h(V^h(\tilde{R}) - \log \hat{c}^0))}}{u_c^h(\hat{c}^h(V^h(\tilde{R})), z_0^h)} \\
    \cdot \left[ \frac{\phi'(\gamma^h (\log \hat{c}^h(V^h(\tilde{R})) - \log \hat{c}^0))}{u_c^h(\hat{c}^h(V^h(\tilde{R})), z_0^h)} D\Delta V^h(\tilde{R}) D\Delta\Delta V^h(\tilde{R}) \right] - \frac{u_c^h(\hat{c}^h(V^h(\tilde{R})), z_0^h)}{u_c^h(\hat{c}^h(V^h(\tilde{R})), z_0^h)} D\Delta V^h(\tilde{R}) D\Delta\Delta V^h(\tilde{R}) + D^2 \Delta \Delta V^h(\tilde{R}) \right]
\]

(156)

where in the last two lines we have used that—by the implicit function theorem—\(\hat{c}'(V^h(\tilde{R})) = u_c^h(\hat{c}^h(V^h(\tilde{R})), z_0^h)^{-1}\).

Using the expressions above, the facts that \(D\Delta w^h(V^h(\tilde{R}))\) and \(D^2 \Delta \Delta w^h(V^h(\tilde{R}))\) are measurable follow from that

- \(\hat{\chi}^h\) and \(\gamma^h\) are measurable (see Appendices E.7.1 and E.7.3),
- \(\hat{c}^h(V^h(\tilde{R}))\) is measurable (see Appendix E.7.4),
- by Assumption 2 the first two derivatives of \(u^h\) in \(c\) and/or \(z\) are measurable,
- the first two Frechet derivatives of \(V^h(\tilde{R})\) are products of the first two derivatives of \(u^h(\hat{c}^h(\tilde{R}), z^h(\tilde{R}))\) (which are measurable by Assumption 2) and the first two Frechet derivatives of \(\hat{c}^h(\tilde{R})\) and \(z^h(\til{R})\) (which are measurable by Lemma 2), and
- products, sums, and (with non-zero denominators) quotient of measurable functions measurable, as are compositions of measurable functions with continuous functions and/or with measurable functions.

It remains to show that \(w^h(V^h(\tilde{R}))\) is measurable. To see this, note that by substituting \(\tilde{u} = V_0^h + \alpha(V^h(\tilde{R}) - V_0^h)\), it may be rewritten as

\[
    w^h(V^h(\tilde{R})) = \chi^h \left( V^h(\tilde{R}) - V_0^h \right) \int_{[0,1]} \frac{e^{\Phi(\gamma^h(V_0^h + \alpha(V^h(\tilde{R}) - V_0^h)) - \log \hat{c}^0))}}{u_c^h(\hat{c}^h(V_0^h + \alpha(V^h(\tilde{R}) - V_0^h)), z_0^h)} d\alpha
\]

(157)

Note that \(V^h(\tilde{R})\) and \(V_0^h\) are measurable as they are the composition of \(u^h(c, z)\) (which is jointly measurable in \((c, z, h)\) with \(\hat{c}^h(\tilde{R})\) and \(z^h(\tilde{R})\) and \(\hat{c}^h(\tilde{R})\) and \(z^h(\tilde{R})\), respectively, all of which are measurable by Lemma 2). This fact, the observation—made in the proof that \(w^h\) is well-defined—that \(V^h(\tilde{R} + B^h(0))\) is convex, and the arguments used above to establish the measurability of \(w^h(V^h(\tilde{R}))\)’s derivatives imply that the integrand in (157) is measurable in \(h\). It remains to argue that the integral in (157) is measurable. To see this note that the integrand is continuous in \(\alpha\); this follows from same the argument used (in the proof that \(w^h\) is well-defined) to argue that the integrand in \(w^h\)’s definition is
continuous. Lemma 4.51 of [Aliprantis and Border 2006] then implies the inner integrand is jointly measurable in \((\alpha, h)\). Finally, the Fubini-Tonelli theorem (as stated in 2.37 of Folland 1999) then implies that the integral of \((157)\) is measurable in \(h\), as desired.

**Proof of regularity, part 1**

We next establish that \(((w^h)_{h \in \mathcal{H}}, G)\) is regular in the sense of Definition[3] Specifically, we establish in this step the existence of integrable functions \(b_0, b_1, b_2 : \mathcal{H} \rightarrow \mathbb{R}\) such that for all \(\tilde{R} \in R + B_\delta(0)\),

\[
\begin{align*}
| (w^h \circ u^h)(c^h(\tilde{R}), z^h(\tilde{R})) | & \leq b_0(h), \\
| c^h(\tilde{R})(w^h \circ u^h)(c^h(\tilde{R}), z^h(\tilde{R})) | & \leq b_1(h), \\
\text{and } | c^h(\tilde{R})^2(w^h \circ u^h)_{\cdot c}(c^h(\tilde{R}), z^h(\tilde{R})) | & \leq b_2(h);
\end{align*}
\]

(158)

We establish the other part of the definition of regularity in the next step.

We begin by proving the bound on the level of welfare using the bounds on the first two derivatives; we then independently establish the bounds on the derivatives. To start, note that by the definition of \(w^h\), \((w^h \circ u^h)(c^h_0, z^h_0) = 0\) for all \(h \in \mathcal{H}\). So applying Taylor’s theorem to the path between \(R\) and any \(\tilde{R} \in R + B_\delta(0)\)—which we may do since \(w^h(V^h(\tilde{R}))\) is twice-continuously Frechet differentiable for \(\tilde{R} \neq R \in R + B_\delta(0)\)—gives us

\[
w^h(V^h(\tilde{R})) = D_{\tilde{R} - R}w^h(V^h(R)) + \frac{1}{2} D^2_{\tilde{R} - R, \tilde{R} - R}w^h(V^h(R + \alpha(\tilde{R} - R)))
\]

(159)

for some \(\alpha \in [0, 1]\), where here we have used that \(\tilde{R} - R \in \Delta\). By the expressions (137) and (138) for the derivatives of \(w^h(V^h(R))\) in the proof of Lemma 5, by Assumptions 1 and 4 by the existence of a uniform bound on \(D \log z^h(\tilde{R})\) (from Lemma 3; see Footnote 60), by the definition of \(\|\cdot\|\), and finally by the existence of the desired bounds on the derivatives of \((w^h \circ c^h)\), there exist constants \(k^{11}_0, k^{21}_0, \) and \(k^{22}_0\) such that, for all \(h \in \mathcal{H}\), \(\tilde{R} \in R + B_\delta(0)\), non-zero \(\Delta \in \Delta\)

\[
\begin{align*}
| D_{\tilde{R} - R}w^h(V^h(R)) | & \leq k^{11}_0 \| \tilde{R} - R \| b_1(h) \\
| D^2_{\tilde{R} - R, \tilde{R} - R}w^h(V^h(R)) | & \leq \| \tilde{R} - R \|^2 \left( k^{21}_0 b_1(h) + k^{22}_0 b_2(h) \right)
\end{align*}
\]

(160)

Combining these bounds with (159), we have

\[
| (w^h \circ u^h)(c^h(\tilde{R}), z^h(\tilde{R})) | \leq \delta k^{11}_0 b_1(h) + \left( k^{21}_0 b_1(h) + k^{22}_0 b_2(h) \right) \frac{\delta^2}{2} \equiv b_0(h)
\]

(161)

We now proceed to the first derivative. From (155), we have

\[
\begin{align*}
| c^h(\tilde{R}) (w^h \circ u^h)_{\cdot c}(c^h(\tilde{R}), z^h(\tilde{R})) | = | c^h(\tilde{R}) u^h(V^h(\tilde{R})) u^h_c(c^h(\tilde{R}), z^h(\tilde{R})) |
\end{align*}
\]

\[
= c^h_0 \delta \left| \frac{c^h(\tilde{R})}{c^h_0} \right| \left| \frac{u^h_c(c^h(\tilde{R}), z^h(\tilde{R}))}{u^h(c^h(\tilde{R}), z^h(\tilde{R}))} \right| e^{\Phi(\log c^h(\tilde{R}) - \log c^h_0)} \leq \varepsilon
\]

(162)

Now note that \(\frac{c^h(\tilde{R})}{c^h_0}\) is uniformly bounded across \(h \in \mathcal{H}\) and \(\tilde{R} \in R + B_\delta(0)\); this follows
from integrating the fact that, by Lemma 3 (see Footnote 60), $|D \log \hat{c}^h(\bar{R})|$ is uniformly bounded. Moreover, $\frac{w^h(c^h(\bar{R}),z^h(\bar{R}))}{u^c(c^h(u),z^h(u))}$ is uniformly bounded by Lemma 11 (see Appendix E.7.4). So there exists some uniform constant $k_1$ such that for all $h \in \mathcal{H}$, $\bar{R} \in R + B_\delta(0)$,

$$\left| c^h(\bar{R})(w^h \circ u^h)_c(c^h(\bar{R}), z^h(\bar{R})) \right| \leq k_1 \hat{\lambda}^h \equiv b_1(h). \quad (163)$$

As we have established in the first step of this proof that $c_0^h \hat{\lambda}^h \equiv b_1(h)$ is integrable, this establishes the desired bound for the first derivative.

Finally, consider the second derivative. We start with the observation that, by (155),

$$(w^h \circ u^h)_c(c^h(\bar{R}), z^h(\bar{R})) = \hat{\lambda}^h \frac{u^h_c(c^h(\bar{R}), z^h(\bar{R}))}{u^h_c(c^h(u^h(\bar{R}), z^h(\bar{R}))), z^h_0)} e^{\Phi(\hat{\gamma}^h(\log \hat{c}^h(u^h(\bar{R}), z^h(\bar{R})) - \log c^h))} \quad (164)$$

An important trick for the remainder of this step is to note that for any differentiable function $\phi^h : V^h(R + B_\delta(0)) \rightarrow \mathbb{R}$ with $\phi^{h'}(u^h(c^h(\bar{R}), z^h(\bar{R}))) > 0$ we have

$$\frac{u^h_c(c^h(\bar{R}), z^h(\bar{R}))}{u^h_c(c^h(u^h(\bar{R}), z^h(\bar{R}))), z^h_0)} = \frac{(\phi^h \circ u^h)_c(c^h(\bar{R}), z^h(\bar{R}))}{(\phi^h \circ u^h)_c(c^h(u^h(\bar{R}), z^h(\bar{R}))), z^h_0)} \quad (165)$$

We will work with the following function $\phi^h$, which (as we will show) has several useful properties:

$$\phi^h(u) = \int_{[u^h_0, u]} \exp \left[ \int_{[u^h_0, u']} \frac{1 - \frac{w^h_u(c^h(\bar{u}), z^h_0)c^h(\bar{u})}{u^h_c(c^h(\bar{u}), z^h_0)} c^h(\bar{u})}{u^h_c(c^h(\bar{u}), z^h_0)} \, d\bar{u} \right] d\bar{u}' \quad (166)$$

To see that $\phi^h(u)$ is well-defined, we first note that (by the same argument used for the well-definedness of $w^h$) $c^h(\cdot)$ is well-defined and strictly positive where evaluated in the definition. That the inner integral used in the definition exists follows from that the integrand is defined—since $c^h(\bar{u}) > 0$ and therefore $u^h_c(c^h(\bar{u}), z^h_0) > 0$ (by Assumptions 2 and 3)—and that the integrand is continuous in $\bar{u}$—since $c^h(\bar{u})$ is continuous in $\bar{u}$ (because $u^h_c > 0$ by Assumption 2) and by Assumption 2 $u^h$ is twice-continuously differentiable. That the outer integral used in the definition exists follows from that the outer integral exists (since the inner integral is defined) and is continuous in $u'$ (since the inner integrand’s continuity allows us to apply the fundamental theorem of calculus).

The same fundamental-theorem-of-calculus arguments also imply that $\phi^h(u)$ is continuously differentiable with

$$\phi^{h'}(u) = \exp \left[ \int_{[u^h_0, u]} \frac{1 - \frac{w^h_u(c^h(\bar{u}), z^h_0)c^h(\bar{u})}{u^h_c(c^h(\bar{u}), z^h_0)} c^h(\bar{u})}{u^h_c(c^h(\bar{u}), z^h_0)} \, d\bar{u} \right] \quad (167)$$

and that $\log \phi^{h'}(u)$ is continuously differentiable with

$$\frac{d}{du} \log \phi^{h'}(u) = 1 - \frac{w^h_u(c^h(u), z^h_0)c^h(u)}{u^h_c(c^h(u), z^h_0)} \quad (168)$$
Next, recall that by Lemma 11 (see E.7.4), there exists \( R \) where the cancellations are by your earlier observation that \( \phi \) is twice-continuously differentiable. Also, by (167), \( \phi^{h'}(u) > 0 \) for all \( u \) in \( \phi^h \)'s domain. Finally, note that for all \( h \in \mathcal{H}, \tilde{R} \in R + B_\delta(0) \),

\[
\left( c^h \circ u^h \right)_{cc} \left( \partial^h (V^h(\tilde{R})), z_0^h \right) \partial^h (V^h(\tilde{R})) \]

\[
= \frac{\partial^{hh'} (V^h(\tilde{R}))}{\phi^{hh'} (V^h(\tilde{R}))} u_c^h \left( \partial^h (V^h(\tilde{R})), z_0^h \right) \partial^h (V^h(\tilde{R})) + \frac{u_c^h \left( \partial^h (V^h(\tilde{R})), z_0^h \right) \partial^h (V^h(\tilde{R}))}{u_c^h \left( \partial^h (V^h(\tilde{R})), z_0^h \right)}
\]

\[
= 1
\]

by (168).

Returning to (164), we now substitute using (165) and differentiate in order to compute \( c^h(\tilde{R})^2 \left( u^h \circ u^h \right)_{cc} (\partial^h (\tilde{R}), z^h(\tilde{R})) \):

\[
\partial^h (\tilde{R})^2 \left( u^h \circ u^h \right)_{cc} (\partial^h (\tilde{R}), z^h(\tilde{R}))
\]

\[
= c_0^h \gamma^h \frac{\partial^h (\tilde{R})}{c_0^h} \left( \phi^h (\tilde{R}) \right) c_{cc}^h \left( \partial^h (V^h(\tilde{R})), z_0^h \right) \partial^h \left( \partial^h (V^h(\tilde{R})) \right)
\]

\[
\left[ \frac{\left( \phi^h (\tilde{R}), z_0^h \right) \partial^h (\tilde{R})}{\left( \phi^h (\tilde{R}), z_0^h \right) \partial^h (\tilde{R})} \right] \partial^h (V^h(\tilde{R})) \partial^h (V^h(\tilde{R}))
\]

\[
= \left( \frac{\phi^h (\tilde{R})}{\phi^h (\tilde{R})} \right) \partial^h (V^h(\tilde{R})) \partial^h (V^h(\tilde{R}))
\]

\[
+ \gamma^h \phi^h \left( \log \partial^h (V^h(\tilde{R})) - \log c^h \right) \frac{\partial^h (\tilde{R})}{\partial^h (V^h(\tilde{R}))} \partial^h (V^h(\tilde{R})) \partial^h (V^h(\tilde{R}))
\]

\[
= \left( \frac{\phi^h (\tilde{R})}{\phi^h (\tilde{R})} \right) \partial^h (V^h(\tilde{R})) \partial^h (V^h(\tilde{R}))
\]

where the cancellations are by our earlier observation that \( \partial^{hh'} (V^h(\tilde{R})) = u_c^h (\partial^h (V^h(\tilde{R})), z_0^h)^{-1} \).

Next, recall that by Lemma 11 (see E.7.4), there exists \( \bar{m} \) such that for all \( h \in \mathcal{H}, \bar{R} \in R + B_\delta(0) \),

\[
\left| \log \partial^h (V^h(\tilde{R})) - \log c^h \left( \bar{R} \right) \right| \leq \bar{m}
\]

\[
\left| \log \left( \left( \phi^h (\tilde{R}) \right) \partial^h (V^h(\tilde{R})), z_0^h \right) \right| - \log \left( \left( \phi^h (\tilde{R}) \right) \partial^h (\bar{R}), z_0^h \right) \right| \leq \bar{m}
\]

and

\[
\frac{d}{d \log c} \left( \phi^h (\tilde{R}) \right) \partial^h (V^h(\tilde{R})), z_0^h \right) - \frac{d}{d \log c} \left( \phi^h (\tilde{R}) \right) \partial^h (\bar{R}), z_0^h \right) \right| \leq \bar{m}.
\]

Also recalling (as argued in the bound of the first derivative), there exists some uniform
bound $\hat{m}$ on $\left| \frac{\epsilon h(\hat{R})}{e_{\epsilon}} \right|$, we may use (170) to bound

\begin{equation}
|c^h(\hat{R})^2 (u^h \circ u^h)^{cc}(c^h(\hat{R}), z^h(\hat{R}))| \leq c_0^h \lambda^h \cdot \hat{m} e^m e^{\Phi} \left[ \bar{m} + \frac{(\phi^h \circ u^h)^{cc}(c^h(V^h(\hat{R})), z_0^h)}{(\phi^h \circ u^h)^{cc}(c^h(V^h(\hat{R})), z_0^h)} c^h(V^h(\hat{R})) \right. \\
+ \left. \frac{(\phi^h \circ u^h)^{cc}(c^h(V^h(\hat{R})), z_0^h)}{(\phi^h \circ u^h)^{cc}(c^h(V^h(\hat{R})), z_0^h)} c^h(V^h(\hat{R})) e^m \right] + |\gamma^h| \Phi e^m \\
= c_0^h \lambda^h \cdot \hat{m} e^m e^{\Phi} \left( \bar{m} + 1 + e^m \right) + |c_0^h \lambda^h \gamma^h| \cdot \hat{m} e^{2m} e^{\Phi} \Phi \equiv b_2(h)
\end{equation}

(172)

which is integrable since we have—in earlier steps of this proof—shown that $c_0^h \lambda^h$ and $c_0^h \lambda^h \gamma^h$ are integrable. To reach the last line we have used that $c_0^h, \lambda^h \geq 0$ and we have used (169).

**Proof of regularity, part 2**

We now establish that $((u^h)_{h \in H}, G)$ satisfies the second component of Definition 3. To do so, we must show that several income-conditional expectations are continuous or continuously differentiable in income.

Since these expectations include products with $\lambda^h(R) \equiv (u^h \circ u^h)^{cc}(c_0^h, z_0^h)$ and $(\lambda \gamma)^h(R) \equiv c_0^h(u^h \circ u^h)^{cc}(c_0^h, z_0^h)$, we first compute expressions of these variables. From (164) and (170), and because $c^h(V^0) = c_0^h$, we have

\begin{equation}
\lambda^h(R) = (u^h \circ u^h)^{cc}(c_0^h, z_0^h) = \tilde{\lambda}^h \quad \text{and} \quad (\lambda \gamma)^h(R) = c_0^h (u^h \circ u^h)^{cc}(c_0^h, z_0^h) = \tilde{\lambda}^h \gamma^h
\end{equation}

(173)

Now, recall from Appendices E.7.2 and E.7.3 the expectations of $R(z_0^h) \tilde{\lambda}^h, R(z_0^h) \tilde{\lambda}^h, R(z_0^h) \tilde{\lambda}^h \gamma^h(R), R(z_0^h) \tilde{\lambda}^h \gamma^h(R), R(z_0^h) \tilde{\lambda}^h \gamma^h(R)$ conditional on income $z_0^h$ exist and are equal to $R(z_0^h) \tilde{\lambda}(z_0^h), R(z_0^h) \tilde{\lambda}(z_0^h) \eta(z_0^h; \epsilon), R(z_0^h) \tilde{\lambda}(z_0^h) \epsilon \leq (z_0^h; \epsilon)$ and $R(z_0^h) \tilde{\lambda}(z_0^h) \left( \frac{z_0^h}{\epsilon} \right) \leq (z_0^h; \epsilon)$,

and $R(z_0^h) \tilde{\lambda}(z_0^h) \left( \frac{z_0^h}{\epsilon} \right)$ respectively (where the moments $x_\leq(z; \epsilon)$ are as in Assumption 6). From our observations in those sections as well as Assumption 6, it moreover follows that $R(z) \tilde{\lambda}(z), R(z) \tilde{\lambda}(z) \epsilon \leq (z; \epsilon)$ and $R(z) \tilde{\lambda}(z) \left( \frac{z}{\epsilon} \right) \leq (z; \epsilon)$, and $R(z) \tilde{\lambda}(z)$ are continuous in $z \in \text{supp} \ g$; and that $R(z) \tilde{\lambda}(z) \eta(z; \epsilon)$ are continuously differentiable in $z \in \text{supp} \ g$.

Combining these observations with (173) implies that expectations of $R(z_0^h) \lambda^h(R), R(z_0^h) \lambda(R) \eta^h(R), R(z_0^h) \lambda(R) \gamma^h(R)$ conditional on income $z_0^h$ exist and are equal to $R(z_0^h) \tilde{\lambda}(z_0^h), R(z_0^h) \tilde{\lambda}(z_0^h) \eta(z_0^h; \epsilon), R(z_0^h) \tilde{\lambda}(z_0^h) \epsilon \leq (z_0^h; \epsilon)$ and $R(z_0^h) \tilde{\lambda}(z_0^h) \left( \frac{z_0^h}{\epsilon} \right) \leq (z_0^h; \epsilon)$, and $R(z_0^h) \tilde{\lambda}(z_0^h) \left( \frac{z_0^h}{\epsilon} \right)$ respectively, and so have the continuity and differentiability properties described above on supp $g$.

**E.7.6 Strengthening Lemmas 6 and 7**

We claim that—under Assumption 4 and for the welfare function defined in the main proof of Theorem 2—Lemmas 6 and 7 hold for all $\Delta \in \Delta$. From the lemmas’ proofs it is clear that we need only show that (i) for all $\Delta \in \Delta$, each additive term of (54), (57), and (55) is integrable over supp $g$ in isolation, and (ii) the integration by parts steps are valid for all $\Delta \in \Delta$ (rather than just the specific $\Delta$s described in the lemmas). To do so, first
we deal with (i) by providing appropriate bounds on each term and second deal with (ii) by (a) providing a general result about integration by parts on \( \text{supp} \, g \), (b) applying it to Lemma 6, and (c) applying it to Lemma 7.

**Bounds on revenue and welfare derivative terms**

We claim that each additive term of the integral expressions in (54), (57), and (65) is integrable in isolation.

By dominated convergence—and since by the definition of \( \Delta \), Assumption (1), and the arguments in the proofs of Lemmas 6 and 7, each term is continuously differentiable and therefore measurable on \( \text{supp} \, g \)—it suffices to show that each is bounded by an integrable function. Because (a) by Assumption 4, \( zg(z) \) and \( R(z)g(z) \) are integrable on \( \text{supp} \, g \), and (b) by the definition of \( \Delta \), \( \frac{\Delta(z)}{R(z)} \), \( \frac{\Delta'(z)}{R(z)} \), and \( \frac{\Delta''(z)}{R(z)} \) are bounded by \( ||\Delta|| \) on \( \text{supp} \, g \), it suffices to show that (i) the terms \( (1-R'(z))z\eta(z) \), \( (1-R'(z))z\epsilon(z) \), and \( \Delta(z) \) from (54), (ii) the terms \( A(z), B(z), C(z), D(z), E(z) \) in (57), and (iii) the terms \( A(z), B(z), C(z) \) in (65) are each bounded by a linear combination of \( z \) and \( R(z) \).

In the case of (54), this is immediate from Assumptions 1, 4, and the definition of \( \Delta \).

In the case of (57), this is immediate from Assumptions 1, 4, and the definition of the conditional expectations used in the definitions of \( A(z), B(z), C(z), D(z), E(z) \).

For the case of (65), we use the observations proven in Appendices E.7.1 and E.7.3 that, over all \( z \in \text{supp} \, g \), \( (\lambda\gamma)(z) = (\lambda\gamma)(z) \) is bounded by a linear combination of \( z \) and \( R(z) \); \( (\lambda \epsilon)(z) = (\lambda \epsilon)(z), (\lambda \eta)(z) = (\lambda \eta)(z), (\lambda \eta)'^2(z) = (\lambda \eta)'^2(z) \leq \epsilon \lambda(z) = \epsilon \lambda(z) \) for \( \epsilon \) the constant defined in the main proof of Theorem 2, and \( \lambda(z) = \hat{\lambda}(z) \) is bounded by a linear combination of \( z \) and \( R(z) \). Combining these observations with Assumption 1 gives the desired bounds.

**General integration by parts on \( \text{supp} \, g \)**

**Lemma 8.** Suppose that \( F(z), G(z) : \text{supp} \, g \rightarrow \mathbb{R} \) are continuously differentiable. Moreover suppose \( F(z) \) and \( G(z) \) are bounded except possibly in limits as \( z \rightarrow 0 \) and/or \( z \rightarrow \infty \) and that for all sequences \( (z_n) \subset \text{supp} \, g \) that converge either to 0 or to \( \infty \), \( \lim_{n \rightarrow \infty} F(z_n)G(z_n)g(z_n) = 0 \). Then if \( F'(z)G(z)g(z) \) and \( F(z)\frac{d(G(z)g(z))}{dz} \) are bounded on \( \text{supp} \, g \) by integrable functions we have

\[
\int_{\text{supp} \, g} F'(z)G(z)g(z)dz = -\int_{\text{supp} \, g} F(z)\frac{d(G(z)g(z))}{dz}dz. \tag{174}
\]

**Proof.** We will use throughout that, by Assumption 5, \( g \) is continuously differentiable.

To start, note that because—by the continuous differentiability of \( F(z), G(z), \) and \( g(z) \) on \( \text{supp} \, g - F'(z)G(z)g(z) \) and \( F(z)\frac{d(G(z)g(z))}{dz} \) are measurable on \( \text{supp} \, g \), and because they are by assumption bounded by integrable functions, they are integrable on \( \text{supp} \, g \) by dominated convergence.

Now, recalling Lemma 9 let \( \mathcal{B} \) be a countable set of disjoint, open, positive intervals,

\[^{108}\]Our ability to interchange \( \lambda \) and \( \hat{\lambda} \) is established in Appendix E.7.5.

\[^{109}\]More formally, for all \( a, b \in \mathbb{R}_0^+ \), \( F(z) \) and \( G(z) \) are bounded within \( \text{supp} \, g \cap [a, b] \).

\[^{110}\]I.e., functions that integrable as random variables on the measure space defined by restricting the standard measure space on \( \mathbb{R} \) to \( \text{supp} \, g \) (which note by Lemma 9 is measurable).
so that \( \text{supp } g = \bigcup_{I \in \mathcal{B}} I \). By the countable additivity of Lebesgue integration,

\[
\int_{\text{supp } g} F'(z)G(z)g(z)\,dz = \sum_{I \in \mathcal{B}} \int_I F'(z)G(z)g(z)\,dz
\]

and

\[
\int_{\text{supp } g} F(z) \frac{d(G(z)g(z))}{dz}\,dz = \sum_{I \in \mathcal{B}} \int_I F(z) \frac{d(G(z)g(z))}{dz}\,dz
\]  

(175)

To complete the proof it therefore suffices to show that for each \( I \in \mathcal{I} \),

\[
\int_I F'(z)G(z)g(z)\,dz = -\int_I F(z) \frac{d(G(z)g(z))}{dz}\,dz.
\]  

(176)

To this end, fix any \( (a, b) \in \mathcal{B} \). For any decreasing sequence \( a_n \to a \) and increasing sequence \( b_n \to b \) so that \( a < a_n < b_n < b \), the fact that \( F(z) \) and \( G(z)g(z) \) are defined and continuously differentiable on \([a_n, b_n]\) implies we may integrate by parts:

\[
\int_{[a_n, b_n]} F'(z)G(z)g(z)\,dz = F(z)G(z)g(z)\bigg|_{a_n}^{b_n} - \int_{[a_n, b_n]} F(z) \frac{d(G(z)g(z))}{dz}\,dz
\]  

(177)

where note the integral on the RHS exists because the integrand is continuous on the closed interval over which it is integrated. Taking the limit as \( a_n \to a \) and \( b_n \to b \), we have by the continuity of Lebesgue integration (and that both integrands are integrable) that

\[
\int_{(a,b)} F'(z)G(z)g(z)\,dz = \lim_{z \to b} F(z)G(z)g(z) - \lim_{z \to a} F(z)G(z)g(z) - \int_{(a,b)} F(z) \frac{d(G(z)g(z))}{dz}\,dz
\]  

(178)

so long as the first two limits on the RHS exist. We begin with the limit from below to \( b \).

If \( b < \infty \), then since \( b > 0 \) by construction, since \( h(b) = 0 \) and \( h \) is continuous, and since by assumption \( F(z)G(z) \) is bounded in the vicinity of \( b \), the limit is zero. Alternatively, if \( b = \infty \), then the limit is zero by assumption. Next, consider the limit from above to \( a \). If \( a > 0 \), then since \( a < \infty \) by construction, the same argument used in the \( b < \infty \) case implies the limit is zero. If instead \( a = 0 \), then the limit is zero by assumption. So both limits exist and equal zero, completing the proof. \( \square \)

**Application to proof of Lemma 6**

It suffices to apply Lemma 8 to each instance of integration by parts in the proof.

We begin with (55). We take \( F(z) = \Delta(z) \) and \( G(z) = g(z) \frac{1 - R'(z)}{R(z)} \varepsilon(z)z \). The proof of Lemma 8 argues that \( F \) and \( G \) are continuously differentiable. That they are bounded except in limits to 0 or \( \infty \) follows from (a) \( \varepsilon(z) \)'s boundedness (Assumption 4), (b) \( g(z) \)'s continuity (Assumption 5), (c) \( R'(z) \)'s continuity (Assumption 1) and (d) \( \Delta \)'s continuity. Now, consider the limit condition: To see it, note that because \( |\Delta(z)| \leq ||\Delta|| R(z) \| \varepsilon(z) \| \), \( \varepsilon(z) \) is bounded, and by Assumption 6 \( \frac{R(z)}{R'(z)} \) is bounded on \( \text{supp } g \), we have that, for all \( z \in \text{supp } g \)

\[
|F(z)G(z)g(z)| \leq k_1 R(z)zg(z) + k_2 z^2 g(z)
\]

for some \( k_1, k_2 \in \mathbb{R}_{\geq 0} \). By Assumption 6 these bounds go to 0 in limits as \( z \to 0 \) or \( z \to \infty \). It remains to show \( F'(z)G(z)g(z) \) and \( F(z) \frac{d(G(z)g(z))}{dz} \) are bounded on \( \text{supp } g \) by integrable functions. To see this, note that by the observations above and Assumptions 4 and 6, there exist
constants $k_n \in \mathbb{R}_{\geq 0}$ such that for all $z \in \text{supp} g$\textsuperscript{111}

$$
|F'(z)(G(z)g(z)| \leq k_3 z g(z) + k_4 R(z)g(z)
$$

$$
F(z) \frac{d(G(z)g(z))}{dz} = \varepsilon(z) \left[ -\frac{1 - R'(z)}{R'(z)} - \alpha(z) + \frac{d}{d \log z} \left( \frac{1 - R'(z)}{R'(z)} \right) + \frac{1 - R'(z)}{R'(z)} \frac{d \log z}{d \log z} \right] \Delta(z)g(z)
$$

$$
\left| F(z) \frac{d(G(z)g(z))}{dz} \right| \leq \left( k_5 + k_6 \frac{1}{R'(z)} \right) R(z)g(z) \leq k_5 R(z)g(z) + k_7 z g(z)
$$

(179)

Note that $z g(z)$ and $R(z)g(z)$ are integrable functions on $\text{supp} g$, since their integrals over $\text{supp} g$ correspond to those of $z_0^1$ and $z_0^0$ over $h \in \mathcal{H}$ and are the latter are integrable by Assumption\textsuperscript{4}.

Second, we consider the $B(z)$ term in (58). We take $F(z) = \frac{\Delta(z)^2}{2}$ and $G(z) = \frac{B(z)}{R(z)R'(z)}$.

Note that $F(z)$ and $G(z)$ are continuously differentiable on $\text{supp} z$ by Assumption\textsuperscript{1} $B(z)$’s continuous differentiability (see the proof of Lemma\textsuperscript{6}), the definition of $\Delta$, and the fact that $R(z)$ and $R'(z)$ are strictly positive on $\mathbb{R}_{>0}$ (see the proof of Lemma\textsuperscript{2}). Next, note that since $R(z)$ and $R'(z)$ are also continuous in $z$ (Assumption\textsuperscript{1}), $R(z), R'(z) > 0$ on $\mathbb{R}_{>0}$, $|\Delta(z)| \leq \|\Delta\| ||R(z)||$, and $B(z)$ is bounded by a linear combination of $z$ and $R(z)$ (shown above), $F(z)$ and $G(z)$ are both bounded on any domain $\text{supp} g \cap [a, b]$, $a, b \in \mathbb{R}_{>0}$.

These observations, combined with the fact that $\frac{R(z)}{R'(z)}$ is bounded on $\text{supp} g$ (Assumption\textsuperscript{6}), imply $|F(z)G(z)g(z)|$ is bounded on $\text{supp} g$ by a linear combination of $z^2 g(z)$ and $z R(z) g(z)$; so Assumption\textsuperscript{6} ensures that for any sequence $(z_n) \subset \text{supp} g$ such that $z_n \to 0$ or $\infty$, we have $\lim_{n \to \infty} F(z_n) G(z_n) g(z_n) = 0$. Differentiating the expression (57) for $B(z)$ reveals that $z B'(z)$ is bounded by $\text{supp} g$ by a linear combination of $z$ and $R(z)$\textsuperscript{112}. This fact, the bound on $B(z)$, the fact that $|\Delta(z)| \leq \|\Delta\| ||R(z)||$ and $|\Delta'(z)| \leq \|\Delta\| ||R'(z)||$, and

\textsuperscript{111}Here, we use that $\alpha(z) \equiv \frac{d \log z g(z)}{d \log z}$ is well-defined since $h$ is differentiable by Assumption\textsuperscript{5} and $g(z) > 0$ on $\text{supp} g$; $\frac{d \log z}{d \log z}$ is well-defined on $\text{supp} g$ since $\varepsilon(z)$ is continuously differentiable by Assumption\textsuperscript{4} and strictly positive by Assumption\textsuperscript{4} and the fact that compensated elasticities are always positive (see the proof of Lemma\textsuperscript{2}).

\textsuperscript{112}More explicitly, we have

$$
x B'(z) = \left[ 2 \left( 1 - R'(z) \right) - 2 x^2 R''(z) \right] \left[ \frac{d \log R(z)}{d \log x} \eta^2(z) + \frac{d \log R'(z)}{d \log x} \eta_1(z) \right] + \frac{d}{d \log x} \left[ \frac{d \log R(z)}{d \log x} \eta^2(z) + \frac{d \log R'(z)}{d \log x} \eta_1(z) \right] \eta(z)
$$

$$
+ \left( \frac{d}{d \log x} \frac{d \log R(z)}{d \log x} \eta^2(z) + \frac{d}{d \log x} \frac{d \log R'(z)}{d \log x} \eta_1(z) \right) \eta(z)
$$

$$
+ \left( \frac{d^2}{d \log x^2} \frac{d \log R(z)}{d \log x} \eta^2(z) + \frac{d^2}{d \log x^2} \frac{d \log R'(z)}{d \log x} \eta_1(z) \right) \eta(z)
$$

$$
- \left( \frac{d}{d \log x} \frac{d \log R(z)}{d \log x} \eta^2(z) + \frac{d}{d \log x} \frac{d \log R'(z)}{d \log x} \eta_1(z) \right) \eta(z)
$$

(180)

By Assumptions\textsuperscript{4,5} and\textsuperscript{6} this implies $z B'(z)$ is bounded by a linear combination of $z$ and $R(z)$.
Assumptions 1 and 6 give us that
\[ |F'(z)(G(z)g(z))| = \left| \Delta(z)\Delta'(z) \frac{B(z)}{R(z)R'(z)} g(z) \right| \leq k_1 z g(z) + k_2 R(z) g(z) \]
and
\[ F(z) \frac{d}{dz} (G(z)g(z)) = \left( \frac{B'(z)}{R'(z)} z + \frac{B(z)}{R(z)R'(z)} \log z \left( \frac{z g(z)}{R(z)R'(z)} \right) \right) \frac{\Delta(z)^2}{2} g(z) \]
\[ = \frac{R(z)}{R'(z)} \left( \frac{B'(z) z}{R'(z)^2} - \frac{B(z)}{R(z)^2} \left( \alpha(z) + 1 + \frac{\log R(z) - \log R'(z)}{d \log z} \right) \right) \frac{\Delta(z)^2}{2} g(z) \]
for all \( z \in \text{supp} \, g \), for various constants \( k_n \in \mathbb{R}_{\geq 0} \). By Assumption 4 these bounds are integrable, as desired.

Third, we consider the \( E(z) \) term in (58). We take \( F(z) = \frac{\Delta'(z)^2}{2} \) and \( G(z) = \frac{z E(z)}{R(z)^2} \). Note that \( F(z) \) and \( G(z) \) are continuously differentiable on \( \text{supp} \, g \) by Assumptions 1, 4, 6, and the definition of \( \Delta \), and the fact that \( R'(z) \) is strictly positive on \( \mathbb{R}_{>0} \) (see the proof of Lemma 2). Since \( R(z) \) and \( R'(z) \) are both continuous in \( z \) (Assumption 3), \( |\Delta'(z)| \leq |\Delta(z)| |R'(z)| \), and \( E(z) \) is bounded by a linear combination of \( z \) and \( R(z) \) (shown above), \( F(z) \) and \( G(z) \) are both bounded on any domain \( \text{supp} \, g \cap [a, b], a, b \in \mathbb{R}_{>0} \). These observations imply \( |F'(z)(G(z)g(z))| \) is bounded on \( \text{supp} \, g \) by a linear combination of \( z^2 g(z) \) and \( z R(z) g(z) \); so Assumption 6 ensures that for any sequence \( (z_n) \subset \text{supp} \, g \) such that \( z_n \to 0 \) or \( \infty \), we have \( \lim_{n \to \infty} F(z_n) G(z_n) g(z_n) = 0 \). Differentiating the expression (57) for \( E(z) \) reveals that \( z E'(z) \) is bounded on \( \text{supp} \, g \) by a linear combination of \( z^2 \) and \( R(z) \). This fact, the bound on \( E(z) \), the fact that \( |\Delta'(z)| \leq |R'(z)| \) and \( |\Delta''(z)| \leq |R'(z)| \), and Assumptions 1 and 6 imply
\[ |F'(z)(G(z)g(z))| = \left| \Delta(z)\Delta''(z) \frac{z E(z)}{R'(z)^2} g(z) \right| \leq k_1 z g(z) + k_2 R(z) g(z) \]
and
\[ F(z) \frac{d}{dz} (G(z)g(z)) = \left( \frac{z E'(z)}{R'(z)^2} + \frac{E(z)}{R'(z)^2} \log z \left( \frac{z g(z)}{R'(z)^2} \right) \right) \frac{\Delta'(z)^2}{2} g(z) \]
\[ = \left( \frac{z E'(z)}{R'(z)^2} - \frac{E(z)}{R'(z)^2} \left( \alpha(z) + 2 \frac{\log R(z) - \log R'(z)}{d \log z} \right) \right) \frac{\Delta'(z)^2}{2} g(z) \]
for all \( z \in \text{supp} \, g \), for various constants \( k_n \in \mathbb{R}_{\geq 0} \). By Assumption 4 these bounds are integrable, as desired.

Fourth, we consider the first term on the RHS of the third line of (58) (the first line proportional to \( D(z) \)). We begin by noting that
\[ \int_{\text{supp} \, g} g(z) D(z) \frac{\Delta(z)\Delta''(z) z}{R(z) R'(z)} dz = \int_{\text{supp} \, g} g(z) \frac{D(z)}{R(z) R'(z)} \left[ \Delta(z) \Delta''(z) + \Delta'(z)^2 \right] dz - \int_{\text{supp} \, g} g(z) D(z) \frac{\Delta'(z)^2}{2} dz \]
\[ = \int_{\frac{1}{\Delta(z)\Delta'(z)}} \right] \left[ \Delta(z) \Delta''(z) + \Delta'(z)^2 \right] dz \]
where the integrals on the RHS by Assumption 1, the definition of \( \Delta \), and the fact shown above that \( D(z) \) is bounded on \( \text{supp} \, g \) by a linear combination of \( z \) and \( R(z) \). We set aside the second term and integrate the first by parts, setting \( F(z) = \Delta(z) \Delta'(z) \) and
\[ E'(z) = 2 (1 - R'(z))(e^2)(z) - 2 z R''(z)(e^2)(z) + 2 z (1 - R'(z))(e^2)'(z) \]
By Assumptions 1, 4, and 6 this implies \( z E'(z) \) is bounded by a linear combination of \( z \) and \( R(z) \).
\( G(z) = \frac{zD(z)}{R(z)R'(z)} \). Note that \( F(z) \) and \( G(z) \) are continuously differentiable on \( \text{supp} \, g \) by Assumption 1, \( D(z) \)'s continuous differentiability (see the proof of Lemma 2), the definition of \( \Delta \), and the fact that \( R(z) \) and \( R'(z) \) are strictly positive on \( \mathbb{R}_{>0} \) (see the proof of Lemma 2). Since \( R(z) \) and \( R'(z) \) are continuous in \( z \) (Assumption 1), \( R(z) \), \( R'(z) > 0 \) on \( \mathbb{R}_{>0} \) (proof of Lemma 2), \( |\Delta(z)| \leq ||\Delta|| \|R(z)\| \) and \( |\Delta'(z)| \leq ||\Delta|| \|R'(z)\| \), and \( D(z) \) is bounded by a linear combination of \( z \) and \( R(z) \), \( F(z) \) and \( G(z) \) are both bounded on any domain \( \text{supp} \, g \cap [a, b] \), \( a, b \in \mathbb{R}_{>0} \). These bounds also ensure \( |F(z)G(z)g(z)| \) is bounded on \( \text{supp} \, g \) by a linear combination of \( z^2g(z) \) and \( zR(z)g(z) \); so by Assumption 6, \( \lim_{n \to \infty} F(z_n)G(z_n)g(z_n) = 0 \) for any sequence \( \{z_n\} \subset \text{supp} \, g \) such that \( z_n \to 0 \) or \( \infty \). Differentiating the expression \( 87 \) for \( \Delta(z) \) reveals that \( \Delta'(z) \) and \( \Delta''(z) \) are bounded by Assumption 1 and \( \Delta'''(z) \) is bounded by Assumption 6.

\[
\begin{align*}
|F'(z)G(z)g(z)| &= \left| \frac{(\Delta(z)\Delta''(z) + \Delta'(z)^2)zD(z)}{R(z)R'(z)} \right| \\
&\leq \left| \frac{\Delta''(z)R'(z)}{R(z)R'(z)} + \Delta'(z)R'(z)R''(z) \frac{D(z)}{R(z)} \right| \leq k_1zg(z) + k_2R(z)g(z) \\
F(z)\frac{d(G(z)g(z))}{dz} &= \frac{D(z)}{R(z)R'(z)} \frac{d\log R(z)}{d\log z} \frac{d\log \left( \frac{zg(z)}{R(z)R'(z)} \right)}{d\log z} \Delta(z)\Delta'(z)g(z) \\
&= \left( \frac{zD'(z)}{R(z)R'(z)} - \frac{D(z)}{R(z)R'(z)} \left( \alpha(z) + \frac{d\log R(z)}{d\log z} + \frac{d\log R'(z)}{d\log z} \right) \right) \Delta(z)\Delta'(z)g(z)
\end{align*}
\]

for all \( z \in \text{supp} \, g \), for various constants \( k_n \in \mathbb{R}_{>0} \). By Assumption 4, these bounds are integrable, as desired.

Fifth, we consider the first term on the RHS of the last line of (58). Take \( F(z) = \frac{\Delta(z)^2}{2} \) and \( G(z) = \frac{1}{g(z)} \frac{d}{dz} \left[ \frac{g(z)zD(z)}{R(z)R'(z)} \right] \). Before beginning to verify the conditions of Lemma 8, it is helpful to note that

\[
G(z) = \frac{1}{R(z)R'(z)} \left[ zD'(z) - D(z) \left( \alpha(z) + \frac{d\log R(z)}{d\log z} + \frac{d\log R'(z)}{d\log z} \right) \right].
\]

Now, note that \( F(z) \) and \( G(z) \) are continuously differentiable on \( \text{supp} \, g \) by Assumptions 4 and 5 (for \( \alpha(z) \)), \( D(z) \)'s twice-continuous differentiability (see the proof of Lemma 2), the definition of \( \Delta \), and the fact that \( R(z) \) and \( R'(z) \) are strictly positive on \( \mathbb{R}_{>0} \) (see the proof of Lemma 2). Since \( R(z) \) is twice-continuously differentiable (Assumption 4), \( R(z) \), \( R'(z) > 0 \) on \( \mathbb{R}_{>0} \) (proof of Lemma 2), \( |\Delta(z)| \leq ||\Delta|| \|R(z)\| \) (definition of \( ||\cdot|| \)), and \( D(z) \) and \( zD'(z) \) are bounded by linear combinations of \( z \) and \( R(z) \) (shown above), \( F(z) \) and \( G(z) \) are both bounded on any domain \( \text{supp} \, g \cap [a, b] \), \( a, b \in \mathbb{R}_{>0} \). Moreover, because \( |\Delta(z)| \leq ||\Delta|| \|R(z)\| \), because \( \frac{d\log R(z)}{d\log z} \) and \( \frac{d\log R'(z)}{d\log z} \) are bounded by Assumption 1.

\[114\]More explicitly, we have

\[
\begin{align*}
D'(z) &= 2(1 - R'(z))(\eta \epsilon)(z) - 2zR''(z)(\eta \epsilon)(z) + 2z(1 - R'(z))(\eta \epsilon)'(z) \\
D''(z) &= -2R''(z)(\eta \epsilon)(z) + 2(1 - R'(z))(\eta \epsilon)'(z) - 2R'(z)(\eta \epsilon)'(z) - 2zR''(z)(\eta \epsilon)'(z) \\
&+ 2(1 - R'(z))(\eta \epsilon)'(z) - 2zR''(z)(\eta \epsilon)'(z) + 2z(1 - R'(z))(\eta \epsilon)''(z)
\end{align*}
\]

By Assumption 4 (and the observation that \( z^2R''(z) = \frac{d^2R(z)}{d\log z^2} - zR''(z) \), Assumption 4 and Assumption 6, this implies \( zD'(z) \) and \( z^2D''(z) \) are bounded by a linear combinations of \( z \) and \( R(z) \).
because $\alpha(z)$ and $\frac{R(z)}{R(z)\prime}$ are bounded on $\text{supp } g$ by Assumption 6, and because $D(z)$ and $zD'(z)$ are bounded by linear combinations of $z$ and $R(z)$, $|F(z)G(z)g(z)|$ is bounded by a linear combination of $z^2g(z)$ and $zR(z)g(z)$ over all $z \in \text{supp } g$. So by Assumption 6, $\lim_{n \to \infty} F(z_n)G(z_n)g(z_n) = 0$ for any sequence $(z_n) \subset \text{supp } g$ such that $z_n \to 0$ or $\infty$. Finally, by our bounding observations on $D(z)$, $zD'(z)$, and $z^2D''(z)$ (see above); the fact that $|\Delta(z)| \leq ||\Delta|| |R(z)|$; and Assumptions 1 and 6, we have

$$
|F(z)G(z)g(z)| \leq k_1 zg(z) + k_2 R(z)g(z)
$$

(done in previous integration-by-parts argument)

$$
F(z) \frac{d}{dz} \left( G(z)g(z) \right) = \frac{\Delta(z)^2}{2} \int \left( \frac{g(z)}{R(z)R'(z)} \left( zD'(z) - \left( \alpha(z) + \frac{d}{d\log z} \log R(z) + \frac{d}{d\log z} \log R'(z) \right) D(z) \right) \right)
$$

for all $z \in \text{supp } g$, for various constants $k_n \in \mathbb{R}_{\geq 0}$. By Assumption 4, these bounds are integrable, as desired.

**Application to proof of Lemma 7**

Consider the term proportional to $(\lambda \eta)(z)$ in (65). Take $F(z) = \frac{\Delta(z)^2}{2}$ and $G(z) = \frac{z(\lambda \eta)(z)}{R(z)}$. Note that $F(z)$ and $G(z)$ are continuously differentiable on $\text{supp } z$ by Assumption 1, the fact that $(wu_h, h \in H, G)$ is a regular social objective (see Appendix E.7.5), the definition of $\Delta$, and the fact that $R(z)$ is strictly positive on $\mathbb{R}_0$ (see the proof of Lemma 2). Next, note that since $R(z)$ is continuous in $z$ (Assumption 1), $R(z) > 0$ on $\mathbb{R}_{\geq 0}, |\Delta(z)| \leq ||\Delta|| |R(z)|$ (definition of $||\Delta||$, $R(z)(\lambda \eta)(z) \leq MR(z)\lambda(z)$ for some constant $M$ (by Assumption 4), and for some $b_1, b_2 > 0, R(z)\lambda(z) \leq b_1 R(z) + b_2 z$ (recalling from Appendix E.7.5) that $\lambda(z) = \bar{\lambda}(z)$, see Appendix E.7.1), we have that $F(z)$ and $G(z)$ are both bounded on any domain $\text{supp } g \cap [a, b], a, b \in \mathbb{R}_{\geq 0}$. Next, note that because $|\Delta(z)| \leq |R(z)|$ and $R(z)(\lambda \eta)(z) \leq MR(z)\lambda(z)$ for some constant $M$, $|F(z)G(z)g(z)|$ is bounded over all $z \in \text{supp } g$ by a constant times $zR(z)\lambda(z)g(z)$; so Assumption 6 ensures that $\lim_{n \to \infty} F(z_n)G(z_n)g(z_n) = 0$ for any sequence $(z_n) \subset \text{supp } g$ such that $z_n \to 0$ or $\infty$. Finally, by Assumptions 1, 4 and 6, and the fact that $|\Delta(z)| \leq ||\Delta|| |R(z)|$ and $|\Delta'(z)| \leq ||\Delta|| |R'(z)|$, we have

$$
|F'(z)G(z)g(z)| = \left| \left( \Delta(z)R'(z) z(\lambda \eta)(z) \right) \frac{R(z)}{R(z)R'(z)} \right| \leq k_1 R(z)\lambda(z)g(z)
$$

$$
F(z) \frac{d}{dz} \left( G(z)g(z) \right) = \left( \frac{z(\lambda \eta)(z)}{R(z)} + \frac{\Delta(z)^2}{2} \right) \left( \frac{zg(z)}{R(z)} \right) \Delta(z)
$$

for all $z \in \text{supp } g$, for various constants $k_n \in \mathbb{R}_{\geq 0}$. Since $R(z)\lambda(z) = R(z)\bar{\lambda}(z)$ is bounded across $\text{supp } g$ by a linear combination of $z$ and $R(z)$ (recalling from Appendix E.7.5) that $\lambda(z) = \bar{\lambda}(z)$, see Appendix E.7.1, these bounds are integrable, as desired.


E.8 Characterization of supp $g$

Lemma 9. supp $g$ is a countable union of disjoint, open, positive intervals.

Proof. First, note that for all $z \in \text{supp } g$, there exists by $h$’s continuity (Assumption 5) some $a_z, b_z \in \mathbb{Q}$ such that $z \in (a_z, b_z) \subset \text{supp } g$. Since $h(0) = 0$ by Assumption 3, we may take $a_z, b_z \geq 0$. Since $(a_z, b_z) \in \mathbb{Q}^2$, which is countable, we conclude that

$$\text{supp } g = \bigcup_{n \in \mathcal{B}} I_n,$$

for $I_n$ positive, open intervals and $\mathcal{B}$ countable.

Next, define an equivalence relation on $\mathcal{B}$ by

$$n \sim m \iff \exists i_1, \ldots, i_k \text{ s.t. } \forall j = 1, \ldots, k - 1, \quad I_{i_j} \cap I_{i_{j+1}} \neq \emptyset$$

Letting $\mathcal{E}$ be the (countable) set of equivalence classes $E$ of $\mathcal{B}$ under $\sim$, we now claim that each union $S_E \equiv \bigcup_{n \in E} I_n$ is an open interval. $S_E$ is open because it is the union of open sets. To see this in turn, suppose not, i.e. $S_E = A \cup B$ with $A, B \neq \emptyset$ and $\overline{A} \cap \overline{B} = A \cap \overline{B} = \emptyset$. This implies that for each $n \in E$, $I_n$ is contained in either $A$ or $B$, since it must be contained in $A \cup B$ and if both $A \cap I_n$ and $B \cap I_n$ are non-empty, then $I_n$ is not connected, contradicting that it is an interval. As a consequence, each $I_{n \in E} \subset A$ can only be in the same equivalence class as other $I_{m \in E} \subset A$ (and similarly for $B$); otherwise, some interval $I_{j \in E}$ on the path between them must contain points in both $A$ and $B$, violating that $\overline{A} \cap \overline{B} = A \cap \overline{B} = \emptyset$. Since $A$ and $B$ are both non-empty and $S_E = A \cup B$, we may therefore take $n, m \in E$ with $I_n \subset A$ and $I_m \subset B$, implying $n \neq m$; this violates $n, m \in E$, a contradiction.

Finally, note that for any distinct $E, E' \in \mathcal{E}$, $S_E$ and $S_{E'}$ are disjoint, since if they intersect then they contain intervals $I_n$ in the same equivalence class. We conclude that

$$\text{supp } g = \bigcup_{n \in \mathcal{B}} I_n = \bigcup_{E \in \mathcal{E}} \bigcup_{n \in E} I_n = \bigcup_{E \in \mathcal{E}} S_E,$$

where we have shown $S_E$ are disjoint, open, positive intervals and $\mathcal{E}$ is countable. \qed

E.9 Indifference curve lemmas

Below, we prove two useful but tedious lemmas that establish uniform bounds on various ratios of derivatives of household utilities. In particular, these lemmas allow us to bound such ratios along households’ indifference curves locally to their consumption-labor profile at the initial equilibrium. These bounds are essential to establishing regularity conditions on the social objective constructed in the proof of sufficiency.

The first lemma below shows that—roughly—given any initial, local tax schedule, there exist other local tax schedules that cause a household to experience any local consumption-labor pair on the same indifference curve as the initial consumption-labor pair.

The second lemma leverages the first to bound ratios of utility derivatives at certain points along indifference curves local to the initial tax schedule.

Lemma 10. There exists $\delta, \bar{\delta} > 0$ with $\delta < \bar{\delta} < \frac{1}{2}$ both small enough that
One may easily verify the following properties of
will use later on.

Preliminary step: a useful deviation

Assumption 4 and Lemmas 2 and 3 apply at $2\delta$ and

for all $h \in \mathcal{H}, \tilde{R} \in R + B_\delta(0), z \in \left[ \min[z_0^h, z^h(\tilde{R})], \max[z_0^h, z^h(\tilde{R})] \right]$, there exists a tax schedule $\tilde{R}^h(\cdot; z, \tilde{R})$ such that

$\cdots$

Proof. We complete the proof in several steps. As a preliminary step, we introduce a convenient tax deviation that will be used throughout. Then we show how to select $\delta$ and $\tilde{\delta}$. We proceed to fix any $h \in \mathcal{H}, \tilde{R} \in R + B_\delta(0)$, and $z \in \left[ \min[z_0^h, z^h(\tilde{R})], \max[z_0^h, z^h(\tilde{R})] \right]$ and then show the existence of an desired tax schedule $\tilde{R}(\cdot; z, \tilde{R})$ by constructing a sequence of tax changes that converge to it.

Preliminary step: a useful deviation

We begin by defining and studying the properties of a useful tax deviation that we will use later on.

For any $\hat{z} \in \mathbb{R}_{>0}$, define $\hat{\Delta}(\cdot; \hat{z}) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by

$$\hat{\Delta}(z; \hat{z}) = \hat{z} \hat{R}'(\hat{z}) p(z; \hat{z})$$

where $p(z; \hat{z}) = \begin{cases} (\log z - (\log \hat{z} - 1))^6 (\log z - \log \hat{z}) (\log z - (\log \hat{z} + 1))^6 & \text{if } z \in [e^{\log z - 1}, e^{\log z + 1}] \\ 0 & \text{else} \end{cases}$

One may easily verify the following properties of $p(\cdot; \hat{z}) : \mathbb{R}_{\geq 0} \to \mathbb{R}$:

- $p(\cdot; \hat{z})$ is three-times continuously differentiable.
- For $n = 0, 1, 2, 3$, the $n^{th}$ log-derivative of $p(\cdot; \hat{z})$—i.e. $\frac{d^n}{d \log z} p(\cdot; \hat{z})$—achieves a maximum absolute value that is independent of $\hat{z}$. Note this implies the existence of $\hat{z}$-independent upper bounds $\overline{p}^1, \overline{p}^2$, and $\overline{p}^3$ on the absolute value of $p'(z; \hat{z})$ $\frac{dp'(z; \hat{z})}{d \log z}$ and $\frac{d^2p'(z; \hat{z})}{d \log z^2}$, respectively.
- $p(\hat{z}; \hat{z}) = 0$ and $p'(\hat{z}; \hat{z}) = \frac{1}{\hat{z}}$

We now consider the implications of these facts for $\hat{\Delta}(\cdot; \hat{z})$. In particular, we claim there exists $\hat{B} > 0$ such that for all $\hat{z} \in \mathbb{R}_{>0}$,

- $\hat{\Delta}(\cdot; \hat{z}) = 0$
- $\hat{\Delta}'(\cdot; \hat{z}) = R'(\hat{z})$
- $\hat{\Delta}(\cdot; \hat{z}) \in \Delta$ and $||\hat{\Delta}(\cdot; \hat{z})|| \leq \hat{B}$

\textsuperscript{115}In particular, this guarantees that labor supply and household utility are well-defined at all tax schedules $\tilde{R} \in R + B_\delta(0)$, which facilitates the rest of the Lemma statement.
The first two bullets are immediate. As our observations on \( p(\cdot; \hat{z}) \) imply \( \hat{\Delta}(\cdot; \hat{z}) \) is three-times continuously differentiable, it remains—in order to prove that \( \hat{\Delta}(\cdot; \hat{z}) \in \Delta \)—to show that \( \exists B \in \mathbb{R} \) s.t. \( \forall z \in \mathbb{R}_{\geq 0} \)

\[
|\hat{\Delta}(z; \hat{z})| \leq B |R(z)|, \quad |\hat{\Delta}'(z; \hat{z})| \leq B |R'(z)|, \quad \left| \frac{d \hat{\Delta}}{d \log z} \right| \leq B |R'(z)|, \quad \text{and} \quad \left| \frac{d^2 \hat{\Delta}}{d \log z^2} \right| \leq B |R'(z)|
\]

(194)

Our observations about the derivatives of \( p(z; \hat{z}) \)—along with the fact that \( p(z; \hat{z}) = 0 \) outside of \( [e^{\log \hat{z} - 1}, e^{\log \hat{z} + 1}] \)—imply that it suffices to show there exists \( B \in \mathbb{R} \) such that (a) in the levels case we have for all \( \log z \in B_1(\log \hat{z}) \), that \( \hat{p}^0 \frac{R'(z)}{R(z)} \hat{z} \leq B R(z) \)

and (b) for \( n = 1, 2, 3 \), we have for all \( \log z \in B_1(\log \hat{z}) \) that \( \hat{p}^n \frac{R'(z)}{R(z)} \hat{z} \leq B R(z) \). To see that such a \( B \) exists note that by Assumption \( 1 \), \( \frac{R(z)}{R'(z)} \frac{R'(z)}{R(z)} \in [e^{-B^R}, e^{B^R}] \) for all \( \log z \in B_1(\log \hat{z}) \); similarly, \( |\frac{\hat{z}}{z}| \leq e^1 \). This implies that we have for \( B = \hat{B} \equiv \max \left[ \hat{p}^0 \frac{R(z)}{R'(z)} \hat{z} \right] \geq B R(z) \)

(194) for \( B = \hat{B} \equiv \max \left[ \hat{p}^0 \frac{R(z)}{R'(z)} \hat{z} \right] \geq B R(z) \). We conclude that for any \( \hat{z} \in \mathbb{R}_{>0}, \Delta(\cdot; \hat{z}) \in \Delta, \) and \( ||\hat{\Delta}(\cdot; \hat{z})|| \leq \hat{B} \).

Choosing appropriate \( \delta, \hat{\delta} \)

We set \( \delta \in (0, \frac{1}{2}) \) small enough that Assumption \( 4 \) and Lemmas \( 2 \) and \( 3 \) apply at \( 2\delta \). Note this guarantees that labor supply \( z^h(\hat{R}) \) and household utility \( V^h(\hat{R}) \) are well-defined and twice-continuously differentiable at all tax schedules \( \hat{R} \in R + \overline{\mathcal{B}}_{\delta}(0) \), which we will use throughout\(^{116}\).

Next, note that by Lemma \( 3 \) (in particular see Footnote \( 60 \)) there exist \( d_z, d_c > 0 \) such that for all \( h \in \mathcal{H}, \hat{R} \in R + \overline{\mathcal{B}}_{\delta}(0) \), and non-zero \( \Delta \in \Delta \),

\[
|D_{\Delta} \log z^h(\hat{R})| \leq d_z ||\Delta|| \quad \text{and} \quad |D_{\Delta} \log c^h(\hat{R})| \leq d_c ||\Delta||
\]

(195)

Note that integrating these bounds imply \( \log z^h(\hat{R}) \in \overline{\mathcal{B}}_{d_z}(\log z^0_h) \) and \( \log c^h(\hat{R}) \in \overline{\mathcal{B}}_{d_c}(\log c^0_h) \); we use these facts later on.

Finally, we take \( \delta > 0 \) small enough that \( \delta + \frac{d_c}{\hat{f}_R} \hat{B} \delta < \hat{\delta} \), where for all \( R > 0 \) is a lower bound on compensated elasticities across all \( h \in \mathcal{H}, \hat{R} \in R + \overline{\mathcal{B}}_{\delta}(0) \); this exists by Assumption \( 4 \).

Setup for main claim

Now fix any \( h \in \mathcal{H}, \hat{R} \in R + \overline{\mathcal{B}}_{\delta}(0) \), and \( z \in \left[ \min[z^0_h, z^h(\hat{R})], \min[z^0_h, z^h(\hat{R})] \right] \). Note that if \( z = z^h(\hat{R}) \), we may set \( \hat{R}^h(\cdot; z, \hat{R}) = \hat{R} \) and we are done. We will therefore prove the claim in the statement of the Lemma assuming \( z^h(\hat{R}) < z \leq z^0_h \); the complementary case where \( z^0_h \leq z < z^h(\hat{R}) \) is analogous.

Before proceeding, recall our goal: Having fixed \( h, \hat{R} \), and \( z \in (z^h(\hat{R}), z^0_h) \), we wish to show the existence of a tax schedule \( \hat{R} \) such that

\[
\hat{R} \in \overline{\mathcal{B}}_{\delta}(0), \quad z^h(\hat{R}) = z, \quad \text{and} \quad V^h(\hat{R}) = V^h(\hat{R}).
\]

(196)

The remainder of the proof shows the existence of \( \hat{R} \) in two basic steps. First, we construct a sequence of tax schedules \( \left( \hat{R}_N \right)_{N \in \mathbb{N}} \) meant to satisfy these conditions in the
limit as $N \to \mathbb{N}$. Second, we argue the limit actually exists and does in fact satisfy the desired conditions.

**Sequence of tax schedules**

As a first step toward constructing the desired sequence of tax schedules $(\tilde{R}_N^N)_{N \in \mathbb{N}}$, define—for every $N \in \mathbb{N}—\epsilon_N \equiv \frac{\log z_0^h - \log z_0^h(\tilde{R})}{N}$ and for all $n = 1, \ldots, N$ and $\epsilon \in [0, \epsilon_N]$, define

$$\tilde{R}_n^N (z; \epsilon) \colon \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad \tilde{z} \mapsto \tilde{R}_n^N (\tilde{z}; \epsilon_N) + \frac{\Delta (\tilde{z}^h(\tilde{R}_n^N (\tilde{z}; \epsilon_N)))}{\Delta (\tilde{z}^h(\tilde{R}_n^N (\tilde{z}; \epsilon_N)))} \epsilon,$$

(197)

where $\tilde{R}_0^N (\cdot; \epsilon_N) \equiv \tilde{R}$. Note that—as can be seen by iterating the definition above and applying the triangle inequality—we have that for all $N \in \mathbb{N}, n = 1, \ldots, N, \epsilon \in [0, \epsilon_N], \tilde{R}_n^N (z; \epsilon) \in R + \mathcal{B}_{\max} (0)$. More strongly, we in fact have, $\tilde{R}_n^N (\cdot; \epsilon) \in R + \mathcal{B}_{\delta} (0)$ since $R \in \mathcal{B}_{\delta} (0)$; since by the definition of $\epsilon_N$, $n \epsilon_N \leq N \epsilon_N \leq \frac{d_{\delta} \delta}{\epsilon / (\epsilon / 2)}$; and since by the definition of $\Delta (\cdot; \tilde{z})$—and our observation in the second proof of this proof that for all $\tilde{R} \in R + \mathcal{B}_{\delta} (0)$, $\log z^h(\tilde{R}) \in \mathcal{B}_{\delta} (0)$—we always have $\tilde{R}_n^N (\tilde{z}; \epsilon) = \tilde{R}(\tilde{z})$ whenever $\log \tilde{z} \not\in [\log z_0^h - d_{\delta} \delta - 1, \log z_0^h + d_{\delta} \delta + 1]$.

We next establish two properties of the tax schedules $\tilde{R}_n^N (\cdot; \epsilon)$ that hold for all sufficiently large $N$. Specifically, for large enough $N$, for all $n = 1, \ldots, N$, $\epsilon \in [0, \epsilon_N]$,

- $\log z^h(\tilde{R}_n^N (\cdot; \epsilon)) \geq \log z^h(\tilde{R}_n^N (\cdot; \epsilon_N)) + \frac{\epsilon^2}{2} \frac{\epsilon}{2}$
- \[|V^h(\tilde{R}_n^N (\cdot; \epsilon)) - V^h(\tilde{R}_n^N (\cdot; \epsilon_N))| \leq \tilde{B} \frac{\epsilon^2}{2} \]

for some constant $\tilde{B}$ independent of $n, N, \epsilon$. Both of these facts are consequences of Taylor’s theorem. More concretely, by the two-times continuous differentiability of $z^h(\cdot)$ and $V^h(\cdot)$ within $R + \mathcal{B}_{\delta} (0)$, we have that—for some $\tilde{c}, \tilde{c} \in [0, \epsilon_N]$—

$$\log z^h(\tilde{R}_n^N (\cdot; \epsilon)) = \log z^h(\tilde{R}_n^N (\cdot; 0)) + D_{\Delta_n} \log z^h(\tilde{R}_n^N (\cdot; 0)) \epsilon + D^2_{\Delta_n} \log z^h(\tilde{R}_n^N (\cdot; \epsilon)) \frac{\epsilon^2}{2}$$

$$\leq \log z^h(\tilde{R}_n^N (\cdot; \epsilon_N)) + \frac{\epsilon}{B} \epsilon - \frac{\epsilon^2}{2}$$

$$V^h(\tilde{R}_n^N (\cdot; \epsilon)) - V^h(\tilde{R}_n^N (\cdot; 0)) = D_{\Delta_n} \frac{\epsilon}{B} V^h(\tilde{R}_n^N (\cdot; 0)) \epsilon + D^2_{\Delta_n} V^h(\tilde{R}_n^N (\cdot; \epsilon)) \frac{\epsilon^2}{2}$$

$$\left| V^h(\tilde{R}_n^N (\cdot; \epsilon)) - V^h(\tilde{R}_n^N (\cdot; \epsilon_N)) \right| \leq 0 + \left( \frac{\epsilon}{B} c_0^h \epsilon + \frac{\epsilon}{B} c_0^h \epsilon + B \frac{1}{\epsilon} (d_1)^2 \frac{\epsilon^2}{2} \right)$$

(198)

where $\Delta_n = \frac{\Delta (z^h(\tilde{R}_n^N (\cdot; 0))))}{\Delta (z^h(\tilde{R}_n^N (\cdot; 0))))}$, where $d_1$ and $d_2$ are upper bounds on the first and second derivatives of $\log$ labor supply across all $\tilde{R} \in \mathcal{B}_{\delta} (0)$ (by Lemma 3 see Footnote 60), and where

$$\pi^h(c, z) \text{ s.t. } \log c \in \mathcal{B}_{\delta} (\log c_0^h), \log z \in \mathcal{B}_{\delta} (\log z_0^h)$$

$$\pi^h(c, z) \text{ s.t. } \log c \in \mathcal{B}_{\delta} (\log c_0^h), \log z \in \mathcal{B}_{\delta} (\log z_0^h)$$

(199)

$^{117}$Note that this justifies our usage of $z^h(\tilde{R}_n^N (\cdot; \epsilon_N))$ in the definition of $\tilde{R}_n^N (\cdot; \epsilon)$.

$^{118}$The maxima defined below exist by Assumption 2.
The first inequality follows from Assumption 4 and the definitions of \( \Delta(\cdot; \tilde{z}, \tilde{R}) \) and \( \hat{B} \) (from the first section of this proof). The second inequality follows from the definition of \( \tilde{\Delta}(\cdot; \tilde{z}, \tilde{R}) \), the expressions (137) and (138) for the first and second derivatives of welfare in the proof of Lemma 5. Assumptions 1 and 4, the fact that \( \tilde{\delta} < \frac{1}{2} \), and the observation (from the second section of this proof) that for all \( \tilde{R} \in \overline{B}_{\tilde{\delta}}(0) \), \( \log z^h(\tilde{R}) \in \overline{B}_{d_{\tilde{\delta}}}(\log z_0^h) \) and \( \log c^h(\tilde{R}) \in \overline{B}_{d_{\tilde{\delta}}}(\log c_0^h) \). (198) implies that claims above (those in bullets) hold so long as \( \epsilon \) is sufficiently small, which—since \( \epsilon \in [0, \epsilon_N] \) and \( \epsilon_N \to 0 \) as \( N \to \infty \)—holds for all \( N \) larger than some \( \tilde{N} \).

We are now almost ready to define \( \tilde{R}^N \), the \( N^{th} \) term of our tax sequence of interest. To do this, note that iterating the first bullet established above implies that, for large enough \( N \), \( \log z^h(\tilde{R}^N_N(\cdot; \epsilon_N)) \geq \log z^h(\tilde{R}) + \frac{\epsilon^2}{\delta} \epsilon_N N = \log z^h(\tilde{R}) \left( \log z_0^h - \log z^h(\tilde{R}) \right) = \log z_0^h \) by the definition of \( \epsilon_N \). Since, for this fixed \( N \), stringing the series of tax schedules \( R^N_N(\cdot; \epsilon) \) generates a continuous path of tax schedules in \( R + \overline{B}_{\tilde{\delta}}(0) \), since \( z^h(\cdot) \) is continuous on this domain, and since \( z \in (z^h(\tilde{R}), z_0^h) \), the intermediate value theorem implies there exists \( n^*(N) \leq N \) and \( \epsilon^*(N) \in [0, \epsilon_N] \) such that \( z^h(\tilde{R}^N_{n^*(N)}(\cdot; \epsilon^*(N))) = z \). We therefore define

\[
\tilde{R}^N = \begin{cases} 
\tilde{R}^N_{n^*(N)}(\cdot; \epsilon^*(N)) & \text{if } N > \tilde{N} \\
\tilde{R} & \text{else.}
\end{cases}
\]

By construction, \( \tilde{R}^N \in R + \overline{B}_{\tilde{\delta}}(0) \). Moreover note that iterating the second bullet proved above implies \( \left| V^h(\tilde{R}^N) - V^h(\tilde{R}) \right| \leq \tilde{B}^N_{\tilde{\delta}} \epsilon_N \to 0 \) as \( N \to \infty \).

**Taking stock**

Let us take stock. So far, we have

- defined some \( \delta, \tilde{\delta} > 0 \),
- fixed arbitrary \( h \in \mathcal{H} \), \( \tilde{R} \in R + B_{\delta}(0) \),
- assumed WLOG that \( z^h(\tilde{R}) < z_0^h \) and fixed \( z \in (z^h(\tilde{R}), z_0^h) \), and
- shown the existence of a sequence of tax schedules \( (\tilde{R}^N)_{N \in \mathbb{N}} \) such that
  - \( \tilde{R}^N \in R + \overline{B}_{\tilde{\delta}}(0) \),
  - \( z^h(\tilde{R}^N) = z \),
  - \( V^h(\tilde{R}^N) \to V^h(\tilde{R}) \) as \( N \to \infty \), and
  - \( \tilde{R}^N(\tilde{z}) = \tilde{R}(\tilde{z}) \) whenever \( \log \tilde{z} \notin B_{\tilde{\delta} + 1}(\log z_0^h) \).

Recall our goal is to show the existence of a tax schedule \( \tilde{R} \) such that

\[
\tilde{R} \in \overline{B}_{\tilde{\delta}}(0), \quad z^h(\tilde{R}) = z, \quad \text{and} \quad V^h(\tilde{R}) = V^h(\tilde{R}).
\]  

In order to do this, the final step of our proof will argue that the sequence \( (\tilde{R}^N)_{N \in \mathbb{N}} \) has a subsequence \( (\tilde{R}^{N_k})_{k \in \mathbb{N}} \) that is Cauchy with respect to the metric \( ||\cdot|| \). Because \( \Delta \) is complete (see Lemma 4), \( R^{N_k} \) converges, and by our earlier observations, converges to some \( \tilde{R} \in \overline{B}_{\tilde{\delta}}(0) \). Finally, the continuity of \( z^h(\cdot) \) implies that \( z^h(\tilde{R}) = z \) and the continuity of \( V^h(\cdot) \) implies \( V^h(\tilde{R}) = V^h(\tilde{R}) \), completing the proof.
Existence of a Cauchy subsequence

Finally, we argue that \( \{\tilde{R}^N\}_{N \in \mathbb{N}} \) has a subsequence which is Cauchy (in the metric \(||\cdot||\)). We will show this by first arguing a subsequence has uniformly convergent third derivatives, and then argue this implies the sequence is Cauchy.

To the point on third derivatives, recall that for all \( N \in \mathbb{N}, \tilde{R}^N(\tilde{z}) = \tilde{R}^N(\tilde{z}) \)—and in particular \( \tilde{R}^N_m(\tilde{z}) = \tilde{R}^m(\tilde{z}) \)—for all \( \tilde{z} \notin \left[ e^{\log z_0^h-d_\delta-1}, e^{\log z_0^h+d_\delta+1} \right] \). So to show that some subsequence has uniformly convergent third derivatives it suffices to do so only at \( \tilde{z} \in \left[ e^{\log z_0^h-d_\delta-1}, e^{\log z_0^h+d_\delta+1} \right] \). On this domain (a finite interval), this is implied by the Arzelà-Ascoli theorem \(^{119}\) applied to the sequence \( \{\tilde{R}^N_m - \tilde{R}^m\}_{N \in \mathbb{N}} \) because

- \( \left| \tilde{R}^N_m(\tilde{z}) - \tilde{R}^m(\tilde{z}) \right| \) is uniformly bounded across \( N \in \mathbb{N}, \tilde{z} \in \left[ e^{\log z_0^h-d_\delta-1}, e^{\log z_0^h+d_\delta+1} \right] \).

To see this, note that since \( \tilde{R}, \tilde{R}^N \in R + \overline{B}_R(0) \), the definition of \(||\cdot||\) implies \( (\tilde{R}^N_m(z) - \tilde{R}^m(\tilde{z}))^2 \in \overline{B}_R(0) \). Because (see the proof of Lemma 2) \( R'(\tilde{z}) > 0 \) for all \( \tilde{z} \in \mathbb{R}_{>0} \) and since \( R'(\tilde{z}) \) is continuous by Assumption 2, it and \( z \) achieve strictly positive upper and lower bounds \( R \) and \( z \) on \( \left[ e^{\log z_0^h-d_\delta-1}, e^{\log z_0^h+d_\delta+1} \right] \). We conclude that \( \left| \tilde{R}^N_m(\tilde{z}) - \tilde{R}^m(\tilde{z}) \right| \leq 4\delta R/\tilde{z} \).

- The sequence \( \{\tilde{R}^N_m(\cdot) - \tilde{R}^m(\cdot)\}_{N \in \mathbb{N}} \) is equicontinuous on \( \left[ e^{\log z_0^h-d_\delta-1}, e^{\log z_0^h+d_\delta+1} \right] \).

To see this, first note that since the function \( \hat{\Delta}(\cdot; \tilde{z}) \) (see the first step of the proof) is four-times continuously differentiable, the construction of \( \tilde{R}^N_m(\cdot) \) implies that \( \tilde{R}^N_m(\cdot) - \tilde{R}^m(\cdot) \) is continuously differentiable. To show equicontinuity, it suffices to show this derivative is uniformly bounded across \( N \in \mathbb{N} \) and \( \tilde{z} \in \left[ e^{\log z_0^h-d_\delta-1}, e^{\log z_0^h+d_\delta+1} \right] \). To see this, note in turn that—again, by the construction of \( \tilde{R}^N \)—

\[
\frac{d}{d\tilde{z}} \left| \tilde{R}^N_m(\tilde{z}) - \tilde{R}^m(\tilde{z}) \right| \leq N\epsilon_N \max_{\tilde{z} \in \left[ e^{\log z_0^h-d_\delta-1}, e^{\log z_0^h+d_\delta+1} \right]} \frac{d^4}{d\tilde{z}^4} \hat{\Delta}(\tilde{z}; \tilde{z})
\]

That the supremum above exists is evident from the definition of \( \hat{\Delta}(\tilde{z}; \tilde{z}) \) in (193). Finally, note that the RHS of the equation above is bounded since \( N\epsilon_N = \log z_0^h - \log z^a(\tilde{R}) < \frac{2\delta}{\hat{\beta}} \)
 which is independent of \( N \) and \( z \).

So far, we have shown the existence of a subsequence \( N_k \) such that \( \tilde{R}^{N_k}_m \) converges to some function—call it \( \tilde{R}_3 : \mathbb{R}_{>0} \to \mathbb{R} \)—uniformly in \( z \). Note that (as each \( \tilde{R}^N_k \) is three-times continuously differentiable) this implies \( \tilde{R}_3 \) is continuous and that \( \tilde{R}_3(\tilde{z}) = \tilde{R}^m(\tilde{z}) \) for \( \tilde{z} \notin \left[ e^{\log z_0^h-d_\delta-1}, e^{\log z_0^h+d_\delta+1} \right] \). Now, define \( \tilde{R}^2 : \mathbb{R}_{>0} \to \mathbb{R} \) by

\[
\tilde{R}^2(\tilde{z}) = \tilde{R}^m(0) + \int_0^{\tilde{z}} \tilde{R}_3(\tilde{z}) d\tilde{z}.
\]

\(^{119}\)The Arzelà-Ascoli theorem is a standard result in functional analysis. It provides conditions under which a sequence of functions has a uniformly convergent subsequence.

\(^{120}\)The definition of \(||\cdot||\) gives us \( \frac{d^2(\tilde{R}^N - R')}{d\log z^a} = z(\tilde{R}^N - R')'(z) \leq 3R'(z) \) and \( \frac{d^4(\tilde{R}^N - R')}{d\log z^a} = z^3(\tilde{R}^N - R')''''(z) \leq 2\delta \). Combining this with the same observation for \( \tilde{R} \) gives us the desired conclusion.
which note is well-defined since $R^3(\tilde{z})$ is continuous. Note that $R^3(\tilde{z}) = R^{2'}(\tilde{z})$. Noting that for each $N_k$,

$$\tilde{R}^{N_k''}(\tilde{z}) = \tilde{R}''(0) + \int_0^\tilde{z} \tilde{R}^{N_k'''}(\tilde{z}) \tilde{z}, \quad (204)$$

we claim that as $k \to \infty$, $\tilde{R}^{N_k''} \to \tilde{R}^2$ uniformly in $\tilde{z}$. To see this fix any $\epsilon > 0$. Since for sufficiently high $k$, $|\tilde{R}^{N_k'''}(\tilde{z}) - \tilde{R}^{3}(\tilde{z})| \leq \frac{\epsilon}{e^{\log z_0^h + d_x \delta - 1} - e^{\log z_0^h - d_x \delta - 1}}$ for all $\tilde{z} \not\in [e^{\log z_0^h - d_x \delta - 1}, e^{\log z_0^h + d_x \delta + 1}]$ and $= 0$ otherwise, we have

$$|\tilde{R}^2(\tilde{z}) - \tilde{R}^{N_k'''}(\tilde{z})| \leq \int_0^\tilde{z} |\tilde{R}^{3}(\tilde{z}) - \tilde{R}^{N_k'''}(\tilde{z})| \tilde{z} \leq \frac{\epsilon}{e^{\log z_0^h + d_x \delta + 1} - e^{\log z_0^h - d_x \delta - 1}} = \epsilon. \quad (205)$$

Repeating this argument again for the first and zeroth derivatives of the subsequence $\tilde{R}^{N_k}$, we have shown the existence of a function $\tilde{R}^0 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that for $n = 0, 1, 2, 3$, $\frac{d^n}{dz^n} \tilde{R}^{N_k} \to \frac{d^n}{dz^n} \tilde{R}^0$ uniformly as $k \to \infty$.

Finally, we wish to conclude that $\tilde{R}^{N_k}$ is Cauchy with respect to the metric $||\cdot||$. To this end, fix $\epsilon > 0$. Take $k$ large enough that for all $k' > k$, $|\tilde{R}^{N_{k'}} - \tilde{R}^0|_{\infty} \leq \tilde{R}_2$, $|\tilde{R}^{N_{k'}} - \tilde{R}^0|_{2} \leq \tilde{R}_2^\frac{1}{2}$, $|\tilde{R}^{N_{k'}} - \tilde{R}^0|_{\infty} \leq \frac{\tilde{R}_2}{\epsilon}$, and $|\tilde{R}^{N_{k'}} - \tilde{R}^0|_{\infty} \leq \frac{\tilde{R}_2}{\epsilon}$, and $|\tilde{R}^{N_{k'}} - \tilde{R}^0|_{\infty} \leq \frac{\tilde{R}_2}{\epsilon}$, where here we have defined $\tilde{R} = \max_{z \in [e^{\log z_0^h - d_x \delta - 1}, e^{\log z_0^h + d_x \delta + 1}]} R(z) > 0$ and $\tilde{R}' = \max_{z \in [e^{\log z_0^h - d_x \delta - 1}, e^{\log z_0^h + d_x \delta + 1}]} R'(z) > 0$; both exist and are strictly positive since $R(\tilde{z}), R'(\tilde{z})$ are continuous and strictly positive for $\tilde{z} \in \mathbb{R}_{\geq 0}$ by Assumption 2 and the proof of Lemma 2. To show that for all $k, k'' > k$, $||\tilde{R}^{N_{k''}} - \tilde{R}^{N_{k'}}|| < \epsilon$, we will (by the triangle inequality) show $||\tilde{R}^{N_{k''}} - \tilde{R}^0|| < \epsilon / 2^{122}$ Indeed, since $\tilde{R}^{(N_{k''})}(\tilde{z}) = \tilde{R}^0(\tilde{z}) = \tilde{R}(\tilde{z})$

\[^{121}]||\cdot||_{\infty} \text{ denotes the sup-norm.}

\[^{122}]\text{Although it is a slight abuse of notation to apply } ||\cdot|| \text{ to } \tilde{R}^{N_{k'}} - \tilde{R}^0 \text{ without having shown that } \tilde{R}^0 \in R + \Delta, \text{ it is easy to see that the argument does not depend on this.}
for all \( z \notin \left[ e^{\log z_0^h - d_\delta -1}, e^{\log z_0^h + d_\delta +1} \right] \)

\[
\left\| \hat{R}^{N'} - \bar{R}^{\delta} \right\| \leq \max \left( \frac{\|\hat{R}^{N'}(\tilde{z}) - \bar{R}^{\delta}(\tilde{z})\|}{|\bar{R}(\tilde{z})|}, \frac{|(\hat{R}^{N'})(\tilde{z}) - (\bar{R}^{\delta})(\tilde{z})|}{|\bar{R}(\tilde{z})|}, \frac{|(\hat{R}^{N'})(\tilde{z}) - (\bar{R}^{\delta})(\tilde{z})|}{|\bar{R}(\tilde{z})|} \right)
\]

\[
= \max \left( \frac{\|\hat{R}^{N'}(\tilde{z}) - \bar{R}^{\delta}(\tilde{z})\|}{|\bar{R}(\tilde{z})|}, \frac{|(\hat{R}^{N'})(\tilde{z}) - (\bar{R}^{\delta})(\tilde{z})|}{|\bar{R}(\tilde{z})|}, \frac{|(\hat{R}^{N'})(\tilde{z}) - (\bar{R}^{\delta})(\tilde{z})|}{|\bar{R}(\tilde{z})|} \right)
\]

\[
\leq \epsilon/2. \tag{206}
\]

**Lemma 11.** There exists \( \delta > 0 \) small enough that the function

\[
\z^h(u) \equiv u^h(\cdot, z_0^h)^{-1}(u) \tag{207}
\]

is, for all \( h \in \mathcal{H} \), well-defined and strictly positive when \( u = V^h(\bar{R}) \) for some \( \bar{R} \in R + B_3(0) \); moreover, \( \z^h(V^h(\bar{R})) \) is \( \mathcal{H} \)-measurable. Further, there exists \( \tilde{m} > 0 \) such that for all \( h \in \mathcal{H}, \bar{R} \in R + B_3(0) \) and for all real-valued functions \( \phi^h \) that are defined and twice differentiable in a neighborhood around \( V^h(\bar{R}) \) and satisfy \( \phi^{h'}(V^h(\bar{R})) > 0 \)—we have \(^{126}\)

\[
\left| \log \z^h \left( V^h(\bar{R}) \right) - \log \z^h \left( \bar{R} \right) \right| \leq \tilde{m}
\]

\[
\left| \log \left( \phi^h \circ u^h \right)_c \left( \z^h(\bar{R}), z_0^h \right) - \log \left( \phi^h \circ u^h \right)_c \left( \z^h(\bar{R}), z_0^h \right) \right| \leq \tilde{m}
\]

and

\[
\left| \frac{d \log}{d \log} \left( \phi^h \circ u^h \right)_c \left( \z^h(\bar{R}), z_0^h \right) - \frac{d \log}{d \log} \left( \phi^h \circ u^h \right)_c \left( \z^h(\bar{R}), z_0^h \right) \right| \leq \tilde{m}. \tag{208}
\]

**Proof.** We complete the proof in three steps. First, we situate the claim in the statement of the Lemma in the context of Lemma 10 and establish the existence of the function \( \z^h(u) \) and the measurability claim; along the way we establish the bound on the levels of log consumption. Second, we prove the bound concerning the first derivatives of \( \phi^h \circ u^h \), and third, we prove the bound concerning the second derivatives of \( \phi^h \circ u^h \).

\(^{126}\)The fact that \( \z^h(V^h(\bar{R})) > 0 \) implies that \( u^h \) twice differentiable and has strictly positive first consumption derivative at all inputs where evaluated above, by Assumption 2, \( \phi^h \) is twice differentiable by assumption. Finally, since \( \phi^{h'}(V^h(\bar{R})) > 0 \) by assumption. Together, these observations imply all derivatives and logs used in the Lemma statement are well-defined.
Note that it suffices to establish the three bounds in (208) for three distinct bounds \( \tilde{m}_1, \tilde{m}_2, \) and \( \tilde{m}_3 \), as we may subsequently take their minimum; we may therefore prove each bound in isolation.

**Indifference curve path**

To begin, take \( \delta \) and \( \delta > \delta \) as in Lemma 10; recall that Assumption 4 and Lemmas 2 and 3 apply \( \hat{\delta} \), so that household labor supply, consumption, and indirect utility are defined at all tax schedules \( \hat{R} \in R + B_{\delta}(0) \). Recall moreover that (from the Lemma) for all \( \hat{R} \in R + B_{\delta}(0), h \in \mathcal{H}, z \in [\min[z^0_h, z^h(\hat{R})], \max[z^0_h, z^h(\hat{R})]], \) there exists \( \hat{R}^h(\cdot; z, \hat{R}) \in R + B_{\delta}(0) \) such that \( z^h(\hat{R}^h(\cdot; z, \hat{R})) = z \) and

\[
V^h(\hat{R}^h(\cdot; z, \hat{R})) = u^h\left(\left(c^h(\hat{R}^h(\cdot; z, \hat{R})), z\right)\right) = V^h(\hat{R}). \tag{209}
\]

We define \( \hat{c}^h(z, \hat{R}) = c^h(R^h(\cdot; z, \hat{R})) \). In particular, note that Note that \( z^h(\cdot, \hat{R}) = u^h(\cdot, z)^{-1}(V^h(\hat{R})) \) by (209)—implying that \( u^h(\cdot, z)^{-1}(V^h(\hat{R})) \) exists—and that \( \hat{c}^h(z, \hat{R}) > 0 \) since it is contained in \( \left[c^h_0 e^{-d_z \delta}, c^h_0 e^{d_z \delta}\right] > 0 \), where \( d_z \) is an upper bound on the first derivative of log labor supply across households and local tax schedules (by Lemma 3; see Footnote 60). Lastly, note that—by the implicit function theorem and since by Assumption 2, \( u^h(\hat{c}^h(z, \hat{R}), z) > 0 \) and \( u^h \) is continuously differentiable—\( \hat{c}^h(z, \hat{R}) \) is continuously differentiable in \( z \in [\min[z^0_h, z^h(\hat{R})], \max[z^0_h, z^h(\hat{R})]] \). Totally differentiating \( u^h(\hat{c}^h(z, \hat{R}), z) = V^h(\hat{R}) \) implies \( c^h(z, \hat{R}) = -\frac{u^h(\hat{c}^h(z, \hat{R}), z)}{u^h(\hat{c}^h(z, \hat{R}), z)} \).

One specific implication of these observations is that the function \( \hat{c}^h(u) \) referred to in (207) of the Lemma statement exists (take \( z = z^0_h \)), satisfies

\[
\log \hat{c}^h(\log(\hat{c}^h)) \in B_{d_z}(\log c^h) \tag{210}
\]

(which note establishes the desired bound in the Lemma statement) and so is strictly positive, and whenever \( u = V^h(\hat{R} \in R + B_{\delta}(0)) \), is equal to \( \hat{c}^h(z^0_h, \hat{R}) \). Note also that (by Assumption 2), \( c^h(z^0_h, \hat{R}) = c^h(\hat{R}) \).

To see that—for any \( \hat{R} \in R + B_{\delta}(0)—\hat{c}^h(\hat{R}) \) is measurable in \( h \in \mathcal{H} \), we apply the measurable maximum theorem as stated in Aliprantis and Border 2006. Specifically, define \( \Gamma : \mathcal{H} \rightrightarrows \mathbb{R}_{>0} \) by \( \Gamma(h) = \left[c^h_0 e^{-d_z \delta}, c^h_0 e^{d_z \delta}\right] \); this is a non-empty and compact-valued correspondence by construction. The fact that \( \Gamma \) is weakly measurable, follows as a special case of the argument made for the correspondence used in the ‘Measurability of labor supply” step of the proof of Lemma 2. Next define \( f : \mathbb{R}_{>0} \times \mathcal{H} \) by \( (c, h) \mapsto \left(u^h(c, z^0_h) - V^h(\hat{R})\right)^2 \). \( f \) is continuous in \( c \) by Assumption 2 and measurable in \( h \) because \( z^0_h \) is by Assumption 3, because \( u^h(\cdot, z^0_h) \) therefore is by Assumption 2 and the composition of measurable functions, and because \( V^h(\hat{R}) = u^h(c^h(\hat{R}), z^h(\hat{R})) \) is by Assumptions 2 and Lemma 2 and the composition of measurable functions. By the measurable maximum theorem, the argmax \( \arg \max_{c \in \Gamma(h)} f(c, h) \) has a measurable selector. However note that \( f(c, h) \) is uniquely maximized by \( \hat{c}^h(V^h(\hat{R})) \), since \( u^h(\hat{c}^h(\log(V^h(\hat{R}))), z^0_h) = \hat{c}^h(V^h(\hat{R})) \).

\footnote{We specialize the theorem to our setting as in the “Measurability of labor supply” step of the proof of Lemma 2. Specifically, we use the following result: If \( \Gamma : \mathcal{H} \rightrightarrows \mathbb{R}_{>0} \) is a weakly measurable correspondence with non-empty compact values and \( f : \mathbb{R}_{>0} \times \mathcal{H} \rightarrow \mathbb{R} \) is a Catheodory function (continuous in its first argument and measurable in its second), then the argmax function \( \mu(h) \equiv \arg \max_{z \in \Gamma(h)} f(z, h) \) admits a measurable selector.}
$V^h(\tilde{R})$ and $u^h$ is strictly increasing in consumption. So $c^h(\tilde{V}^h(\tilde{R}))$ is measurable in $h$.

**First derivative bounds**

We now consider the first bound in (208). To start, note that for any $h \in \mathcal{H}, \tilde{R} \in R + B_\delta(0)$, $z \in [\min[z^h, z^h(\tilde{R})], \max[z^h, z^h(\tilde{R})]]$, we have

$$
\frac{d}{dz} \log \left( \phi^h \circ u^h \right) \left( c^h(z, \tilde{R}), z \right) = \frac{\left( \phi^h \circ u^h \right) c \left( c^h(z, \tilde{R}), z \right)}{\phi^h \circ u^h} \left( c^h(z, \tilde{R}), z \right) + \frac{\left( \phi^h \circ u^h \right) c \left( c^h(z, \tilde{R}), z \right)}{\phi^h \circ u^h} \left( c^h(z, \tilde{R}), z \right)
$$

Next—for any $h \in \mathcal{H}, \tilde{R} \in R + B_\delta(0)$, by Assumption 2, where we have used that $M^h(c, z) = \frac{u^h_c(c, z)}{\log_{\phi^h \circ u^h}(c, z)}$, where we have used the definition of $\eta^h$ and $c^h$ in (212), where $\eta$ and $\varepsilon > 0$ are upper and lower bounds on the magnitude of elasticities—per Assumption 4 and the fact that all $\tilde{R}(.; z, \tilde{R}) \in R + B_\delta(0)$—, and where the cancellation is since, by the design of the path,

$$
\frac{d}{dz} \left( \phi^h \circ u^h \right) \left( c^h(z, \tilde{R}), z \right) = \left( \phi^h \circ u^h \right) c \left( c^h(z, \tilde{R}), z \right) = 0.
$$

Next—for any $h \in \mathcal{H}, \tilde{R} \in R + B_\delta(0)$—the continuous differentiability of $c^h(\cdot, \tilde{R})$ and $\phi^h \circ u^h$ (by the conditions of the Lemma and Assumption 2) allow us to apply the fundamental theorem of calculus:

$$
\left| \log \left( \phi \circ u \right) \left( c^h(V^h(\tilde{R})), z_0 \right) - \log \left( \phi \circ u \right) \left( c^h(\tilde{R}), z_0 \right) \right| = \left| \log \left( \phi \circ u \right) \left( c^h(\tilde{R}), z_0 \right) - \log \left( \phi \circ u \right) \left( c^h(\tilde{R}), z_0 \right) \right|
$$

$$
= \left| \max \left[ \log z_0, \log z \right] \int \left( \phi \circ u \right) \left( c^h(\tilde{R}), z \right) d\log z \right|
$$

$$
\leq \frac{\eta}{\varepsilon} \int \left| \frac{d}{d\log z} \left( \phi \circ u \right) \left( c^h(\tilde{R}), z \right) \right| d\log z = \frac{\eta}{\varepsilon} \int \left( \phi \circ u \right) \left( c^h(\tilde{R}), z \right) d\log z
$$

In the second-to-last step, we have used that $c^h(\tilde{R})$ is increasing in $z$, by Assumption
In the final step, $d_c$—and so the entire bound—is constant across all $h \in \mathcal{H}$ and $\tilde{R} \in R + B_\delta(0)$ (by Lemma 3, see Footnote 50), and we have used that $\tilde{R}^h(\cdot; z^h_0, \tilde{R}), \tilde{R} \in R + B_\delta(0)$.

**Second derivative bounds**

Finally, we consider the second bound in (208). To start, note that for any $h \in \mathcal{H}, \tilde{R} \in R + B_\delta(0), z \in [\min[z^h_0, z^h(\tilde{R})], \max[z^h_0, z^h(\tilde{R})]],$ we have

\[
\frac{d}{dz} \left( \phi^h \circ u^h \right)_c \left( \phi^h(z, \tilde{R}), z \right) \phi^h(z, \tilde{R}) = \left( \frac{\phi^h \circ u^h}{\phi^h \circ u^h}_c \left( \phi^h(z, \tilde{R}), z \right) \phi^h(z, \tilde{R})^2 \right) \frac{\phi^h(\tilde{R}, z)}{\phi^h(z, \tilde{R})} + \left( \frac{\phi^h \circ u^h}{\phi^h \circ u^h}_c \left( \phi^h(z, \tilde{R}), z \right) \phi^h(z, \tilde{R}) \right) \phi^h(z, \tilde{R})^2 \frac{\phi^h(\tilde{R}, z)}{\phi^h(z, \tilde{R})^2} \left( \phi^h \circ u^h \right)_c \left( \phi^h(z, \tilde{R}), z \right) \frac{\phi^h(\tilde{R}, z)}{\phi^h(z, \tilde{R})} \frac{\phi^h(\tilde{R}, z)}{\phi^h(z, \tilde{R})}
\]

where above we have again made use of the fact that $M^h(c, z) = -\frac{u^h_c(z)}{u^h(z)} = \left( \frac{\phi^h \circ u^h}{\phi^h \circ u^h}_c \right)_c (c, z)$

Next, we use the facts that

\[
\frac{d}{dz} \frac{\log M^h \left( \phi^h(\tilde{R}, z), z \right)}{\phi^h(z, \tilde{R})} = -\frac{\eta^h(\tilde{R} : z, z)}{\phi^h(z, \tilde{R})},
\]

and

\[
\frac{d}{dz} \frac{\log M^h \left( \phi^h(\tilde{R}, z), z \right)}{\phi^h(z, \tilde{R})} = -\frac{\eta^h(\tilde{R} : z, z)}{\phi^h(z, \tilde{R})} + \frac{2 \eta^h(\tilde{R} : z, \tilde{R}) \phi^h(\tilde{R}, z) \phi^h(z, \tilde{R}) - \eta^h_0(\tilde{R} : z, z) \phi^h(\tilde{R}, z) \phi^h(z, \tilde{R})}{\phi^h(z, \tilde{R})^2} \phi^h_+ \left( \tilde{R} : z, z \right)
\]

which are both easily verified from the formulae (121), (125), and (127) for elasticities.
and super-elasticities in the proof of Lemma 2. Note that by \( \check{c}^h(z, \tilde{R}) \)'s continuity in \( z \) and Assumption 2, both terms are continuous in \( z \).

Finally—since by \( \check{c}^h(z, \tilde{R}) \)'s continuity in \( z \), \( \phi^h \)'s twice-continuous differentiability, and Assumption 2 \( \frac{d}{d \log z} \frac{\check{c}^h(V^h(\tilde{R}), z)}{(\phi^h \circ u^h)_c(\check{c}^h(V^h(\tilde{R}), z))} \) is continuous in \( z \)—we may integrate:

\[
\int_{\log z_0^h}^{\log \hat{z}^h(\tilde{R})} \frac{\hat{z}^h(\tilde{R})}{d \log z} \left( \frac{d \log M^h(\check{c}^h(z, \tilde{R}), z)}{d \log c} \right) d \log \check{c}^h(z, \tilde{R}) + \left( \frac{\eta^h(\tilde{R} ; z, \tilde{R})}{\check{h}^h(z, \tilde{R})} - 1 \right) \frac{d \log (\phi^h \circ u^h)_c(\check{c}^h(z, \tilde{R}), z)}{d \log z} d \log z
\]

where \( \eta_{z+0}, \bar{z}_{z+0}, \bar{z}_{z+1}, \eta, \) and \( \xi \) are upper and lower bounds (as indicated by the notation) of the corresponding super-elasticities and elasticities, per Assumption 4 and the fact that all \( \tilde{R}(\cdot; z, \tilde{R}) \in R + B_{\beta}(0) \); and where above we have used the fact that \( \check{c}^h(z, \tilde{R}) \) is increasing in \( z \), by its definition and Assumption 2.

Since we have already shown the last term is uniformly bounded across \( h \in \mathcal{H} \) and \( \tilde{R} \in R + B_{\beta}(0) \), we have the desired conclusion. \( \square \)

F Additional Tables and Figures
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Figure 38: DEFG test evaluated from 1979 (top left), to 1982 (bottom right), with and without the two final terms of (20). 90% confidence bands
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Figure 40: DEFG test evaluated from 1987 (top left), to 1990 (bottom right), with and without the two final terms of (20). 90% confidence bands